Chapter one

"Random Variables"

1– Th concept of Random variables مفهوم المتغيرات العشوائية

Definitions:

Given a random experiment with sample space $\boldsymbol{S}$, A random variable (r.v.) X is a function which assigns to each element $\boldsymbol{w} \in \boldsymbol{S}$ a real number $\boldsymbol{X}(\boldsymbol{w})$ in the set $\boldsymbol{E}$.

Ex) a _ Suppose that an experiment consists of tossing two fair coins :

SoL) $S = \{HH, HT, TH, TT\}$ _____ random experiment

Random variables:

- X : تمثل عدد مرات ظهور (H)
  X= 0 , 1 , 2
- Y : تمثل عدد مرات ظهور (T)
  Y= 0 , 1 , 2

$\boldsymbol{P(X = x)}$

$$\boldsymbol{P(X=0) = \frac{1}{4}}$$

$$\boldsymbol{P(X=1) = \frac{2}{4}}$$

$$\boldsymbol{P(X=2) = \frac{1}{4}}$$

$$\longrightarrow \sum \boldsymbol{P(X = x_i) = 1}$$

$\underline{\boldsymbol{X}}$ **Values**

$\boldsymbol{HH \to 2}$

$\boldsymbol{HT \to 1}$

$\boldsymbol{TH \to 1}$

$\boldsymbol{TT \to 0}$

b _ Suppose that an experiment consists of tossing three fair coins :-

Sol/

$$S = \{\underbrace{HHH}_{3}, \underbrace{HHT, HTH, THH}_{2}, \underbrace{TTH, THT, HTT}_{1}, \underbrace{TTT}_{0}\}$$

X = 0 , 1 , 2 , 3 X تمثل عدد مرات ظهور الصورة (H)

$$P(X=0) = 1/8$$

$$P(X=1) = 3/8$$

$$P(X=2) = 3/8$$

$$P(X=3) = 1/8$$

$$\sum p(x) = 1$$

$3 \leftarrow HHH$

$2 \leftarrow \{HHT, HTH, THH\}$

$1 \leftarrow \{TTH, THT, HTT\}$

$0 \leftarrow TTT$

c _ Suppose that an experiment consists of tossing a pair of unbiased dice , X be a random variable indicating the sum of the two faces :-

Sol/

X =	2	3	4	5	6	7	8	9	10	11	12
	(1,1)	(1,2)	(1,3)	(1,4)	(15)	(16)	(26)	(36)	(46)	(56)	(66)
		(21)	(22)	(23)	(24)	(25)	(35)	(45)	(55)	(65)	
			(31)	(32)	(33)	(34)	(44)	(54)	(64)		
				(41)	(42)	(43)	(53)	(63)			
					(51)	(52)	(62)				
						(6,1)					

$$P(X=x)=\frac{1}{36},\frac{2}{36},\frac{3}{36},\frac{4}{36},\frac{5}{36},\frac{6}{36},\frac{5}{36},\frac{4}{36},\frac{3}{36},\frac{2}{36},\frac{1}{36}$$

X	2	3	4	5	6	7	8	9	10	11	12
$P(X=x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

And $\sum P(X=x_i)=1$

ويمكن تمثيل (r.v.) وفق دوال وعلى النحو الاتي :-

$$P(X)\begin{cases} 0 & x=0 \\ \frac{X-1}{36} & x=2,3,4,5,6,7 \\ \frac{13-X}{36} & x=8,9,10,11,12 \\ 1 & \Sigma P(X)=1 \end{cases}$$

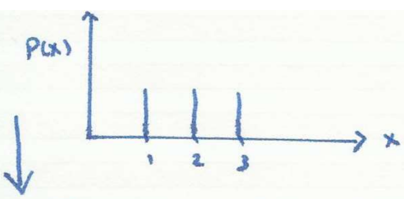
ويمكن تمثيل دالة خطية عند رمي زار مرة واحدة

$S = \{1, 2, 3, 4, 5, 6\}$

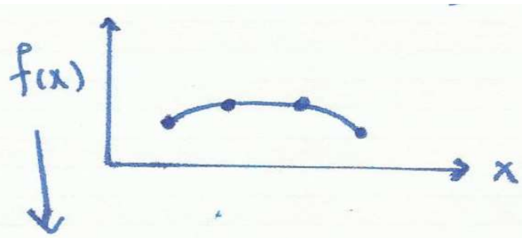
$$P(x) = \begin{cases} \frac{1}{6} & x = 1, 2, 3, \dots, 6 \\ 0 & \text{other wise} \end{cases}$$

Random Variable (r.v.)

متقطع discretes (r.v.) $x = 0, 1, 2, 3, 4, \dots$	مستمر Continuous (r.v.) $a < x < b$
Probability mass function دالة كتلة احتمالية (P.m.f.) رمزها P(x) في اغلب الاحيان شكلها	Probability density function دالة كثافة احتمالية (P.d.f.) رمزها f(x) شكلها



خواص الدالة



1- $P(x) \geq 0$; $0 \leq P(X_i) \leq 1$, $i = 1, 2, 3, \dots$

2- $\sum_{i=1}^{n} P(X_i) = 1$

$f(x) \geq 0$; $0 \leq f(x) \leq 1$

$\int_{-\infty}^{\infty} f(x)\, dx = 1$

دالة الاحتمال التراكمية

Cumulative Probability function

$F(x) = \sum_{X_i < x} P(X_i)$ | $F(x) = \int_{-\infty}^{x} f(x)\, dx$

2– Distribution Function (D.F.) دالة التوزيع

Definitions:

If X is a random variable (r.v.) defined on the sample space S , the cumulative distribution function (c. d. f.) or cumulative probability function (c. p. f.) of a discrete random variable , denoted by $F(x)$ & is defined by :-

$$F(x) = P(X \leq x) \quad \text{for any real number}$$

$$= \sum P(x_i)$$

Properties :

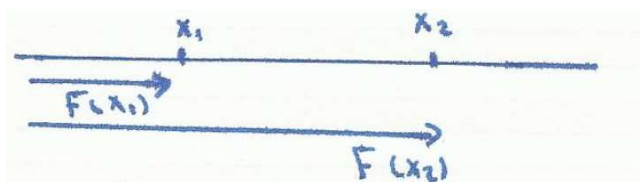
The distribution function $F(x)$ has following properties :-

1_ $F(\infty) = \lim_{x\to\infty} F(x) = 1$

2_ $F(-\infty) = \lim_{x\to-\infty} F(x) = 0$

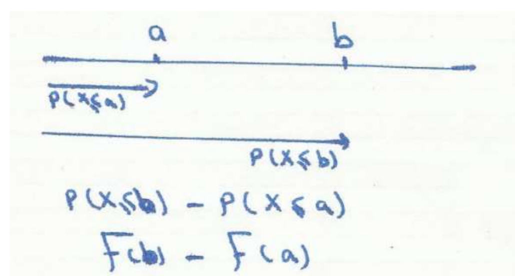
$0 \leq F(x) \leq \mathbf{1}$

3_ $F(x_1) \leq F(x_2)$; $if\ \ x_1 \leq x_2$



4_ $\boldsymbol{P(a < x \leq b)} = F(b) - F(a) \quad for\ all \quad a < b$

$\boldsymbol{P(x > b)} = 1 - F(b)$



Ex)

$$\boldsymbol{P(x)} = \begin{cases} \frac{x}{6} & , \quad \boldsymbol{x = 1, 2, 3} \\ \mathbf{0} & , \quad other\ wise \end{cases}$$

Find :

1_ Probability mass function

2_ cumulative probability function

Sol) 1-

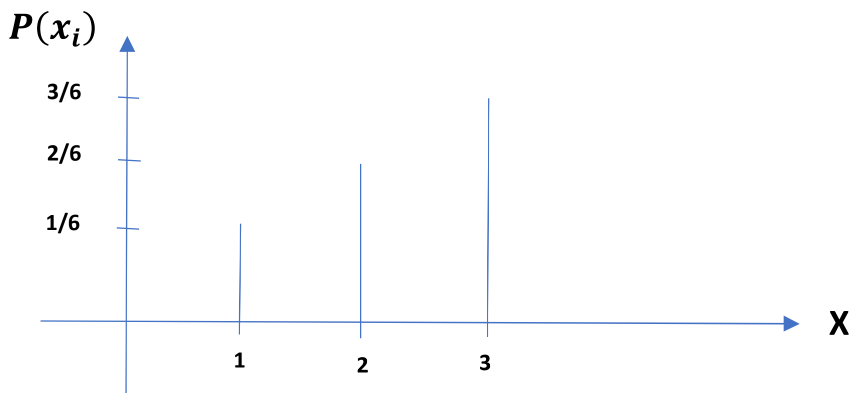
x	1	2	3
$P(X=x)$	1/6	2/6	3/6

$$\sum P(X=x_i)=1$$

$$\sum P(X=x_i)= P(X=1)+ P(X=2)+ P(X=3)=\frac{1}{6}+\frac{2}{6}+\frac{3}{6}=\frac{6}{6}=1$$

تحقق الشرط

رسم الدالة



2_ $F(X)=P(X\le x)$

$$F(0)=P(X\le 0)= P(X=0)=0$$

$$F(1)=P(X\le 1)= P(X=0)+ P(X=1) =\frac{1}{6}$$

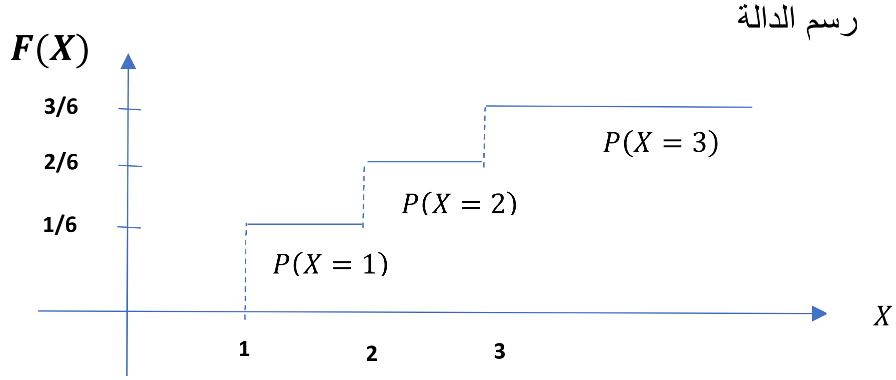
$$F(2)=P(X\le 2)= P(X=0)+ P(X=1) + P(X=2)=\frac{1}{6}+\frac{2}{6}=\frac{3}{6}$$

$$F(3)=P(X\le 3)= P(X=0)+ P(X=1) + P(X=2)+P(X=3)$$

$$=\frac{1}{6}+\frac{2}{6}+\frac{3}{6}=\frac{6}{6}=1$$

$$\therefore F(X)=\begin{cases} 0 & x<1 \\ \frac{1}{6} & 1\le x<2 \\ \frac{3}{6} & 2\le x<3 \\ 1 & x\ge 3 \end{cases}$$

رسم الدالة



وفي حالة توفر الدالة التراكمية فقط والمطلوب إيجاد دالة الكتلة الاحتمالية $P(X)$ يتم اجراء عملة الطرح من الأعلى الى الأسفل وفق الأسلوب الاتي :-

_ $P(0 < x \leq 1)$

$$P(X = 1) = F(b) - F(a) = \frac{1}{6} - 0 = \frac{1}{6} \qquad P(a < x \leq b)$$

_ $P(1 < x \leq 2)$

$$P(X = 2) = F(2) - F(1) = \frac{3}{6} - \frac{1}{6} = \frac{2}{6}$$

_ $P(2 < x \leq 3)$

$$P(X = 3) = F(3) - F(2) = 1 - \frac{3}{6} = \frac{3}{6}$$

Ex) Two fair dice are thrown once , let X denoted the maximum of the two number shown by two dice , then the *P.m.f.* of X is given by :-

$\boldsymbol{x}$	1	2	3	4	5	6
$\boldsymbol{P(X = x)}$	$1/36$	$3/36$	$5/36$	$7/36$	$9/36$	$11/36$

SoL)

$F(x) = P(X \leq x) = P(X = ?)$

$F(0) = P(X \leq 0) = P(X = 0) = 0$

$F(1) = P(X \leq 1) = P(X = 1) = \frac{1}{36}$

$F(2) = P(X \leq 2) = P(X = 1) + P(X = 2) = \frac{1}{36} + \frac{3}{36} = \frac{4}{36}$

$F(3) = P(X \leq 3) = P(X = 1) + P(X = 2) + P(X = 3) = \frac{1}{36} + \frac{3}{36} + \frac{5}{36} = \frac{9}{36}$

$F(4) = P(X \leq 4) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) = \frac{16}{36}$

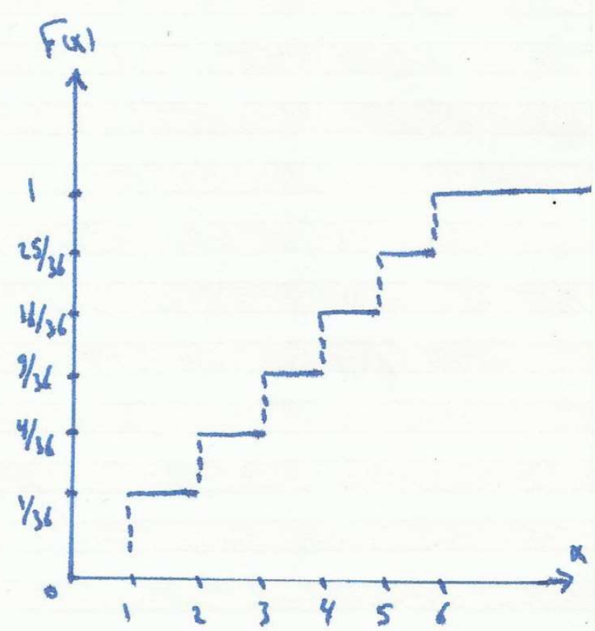
$F(5) = P(X \leq 5) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) = \frac{25}{36}$

$F(6) = P(X \leq 6) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6) = \frac{36}{36} = 1$

$F(7) = 1 \quad , F(8) = 1$

1	2	3	4	5	6
(11)	(22)	(33)	(44)	(55)	(66)
	(12)	(23)	(34)	(45)	(56)
	(21)	(32)	(43)	(54)	(65)
		(13)	(24)		
		(31)	(42)		
			(14)		
			(41)		

$$F(x) = \begin{cases} 0 & x < 1 \\ 1/36 & 1 \leq x < 2 \\ 4/36 & 2 \leq x < 3 \\ 9/36 & 3 \leq x < 4 \\ 16/36 & 4 \leq x < 5 \\ 25/36 & 5 \leq x < 6 \\ 1 & x \geq 6 \end{cases}$$



Ex) A box contains 6 red and 4 whit balls. Two balls are drawn at random , Let Y denote the number or red balls that can be drawn . Then Y must assume one of the value 0 , 1 and 2 with respective probability 2/15 , 8/15 and 5/15 , Therefore , the (P. m. f.) of Y is given by :-

$\boldsymbol{y}$	0	1	2
$\boldsymbol{P(Y=y)}$	$2/15$	$8/15$	$5/15$

$$\sum \boldsymbol{P(Y=y_i) = P(Y=0) + P(Y=1) + P(Y=2) = \frac{2}{15} + \frac{8}{15} + \frac{5}{15} = \frac{15}{15} = 1}$$

The distribution function (c.p.f.) or (D.F.)

$F(y) = P(Y \le y)$

$F(0) = P(Y < 0)$

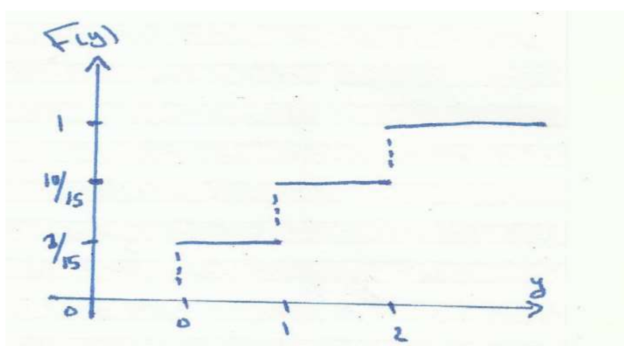
$$F(0) = P(Y \le 0) = P(Y=0) = \frac{2}{15}$$

$$F(1) = P(Y \le 1) = P(Y=0) + P(Y=1) = \frac{2}{15} + \frac{8}{15} = \frac{10}{15}$$

$$F(2) = P(Y \le 2) = P(Y=0) + P(Y=1) + P(Y=2) = \frac{10}{15} + \frac{5}{15} = \frac{15}{15} = 1$$

$F(3) = P(Y \le 3) = 1$

$$F(y) = \begin{cases} 0 & x < 0 \\ 2/15 & 0 \le x < 1 \\ 10/15 & 1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$



اذا طلب دالة $P(y)$ عند توفر دالة $F(y)$ فإن

$$P(a < y \le b) = F(b) - F(a)$$

$P(y = b)$

$$P(0 < y \le 1) = F(1) - F(0) = \frac{10}{15} - \frac{2}{15} = \frac{8}{15}$$

$P(y = 1)$

$$P(1 < y \leq 2) = F(2) - F(1) = 1 - \frac{10}{15} = \frac{5}{15}$$

$$P(y = 2)$$

Ex) Three balls are chosen at random out of (5) numbered ball from 2 to 6 , Let X be the sum of the three balls . Construct P.m.f. of X and F(x) .

Sol)

$$2 \ , \ 3 \ , \ 4 \ , \ 5 \ , \ 6 \quad \therefore \ n = 5 \ , \ r = 3$$

$$C_r^n = \frac{n!}{r!\,(n-r)!} = \frac{5!}{3!\,2!} = 10 \ \ way$$

$$S = \{234\,, 235\,, 236\,, 245\,, 246\,, 256\,, 345\,, 346\,, 356\,, 456\,\}$$

Sum = 9 10 11 11 12 13 12 13 14 15

$\boldsymbol{x}$	9	10	11	12	13	14	15
$\boldsymbol{P(X = x)}$	1/10	1/10	2/10	2/10	2/10	1/10	1/10

$$\sum_{i=9}^{15} P(x_i) = \frac{1}{10} + \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{2}{10} + \frac{1}{10} + \frac{1}{10} = \frac{10}{10} = 1$$

$$F(x) = P(X \leq x)$$

$$F(x) = \begin{cases} 0 & x < 9 \\ 1/10 & 9 \leq x < 10 \\ 2/10 & 10 \leq x < 11 \\ 4/10 & 11 \leq x < 12 \\ 6/10 & 12 \leq x < 13 \\ 8/10 & 13 \leq x < 14 \\ 9/10 & 14 \leq x < 15 \\ 1 & x \geq 15 \end{cases}$$

Ex) Let X be a random variable (r.v.) with the following distribution . Find the cumulative distribution function(C.D.F.) of X and draw the diagram .

$$P(X) = \begin{cases} \frac{1}{4} & x = -2 \\ \frac{1}{8} & x = 1 \\ \frac{1}{2} & x = 2 \\ \frac{1}{8} & x = 4 \\ 0 & other\ wise \end{cases}$$

Sol)

$$F(x) = P(X \le x)$$

$$\therefore F(x = -3) = P(X \le -3) = P(X = -3) = 0$$

$$F(x = -2) = P(X \le -2) = P(X = -2) = \frac{1}{4}$$

$$F(x = 1) = P(X \le 1) = P(X = -2) + P(X = 1) = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$$

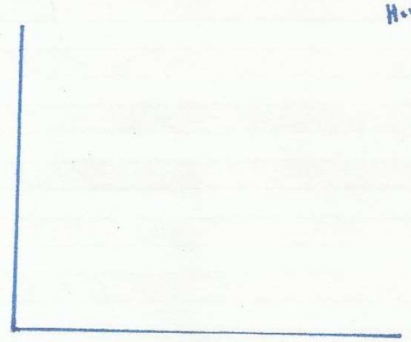
$$F(x = 2) = P(X \le 2) = P(X = -2) + P(X = 1) + P(X = 2) = \frac{3}{8} + \frac{1}{2} = \frac{7}{8}$$

$$F(x = 4) = P(X \le 4) = P(X = -2) + P(X = 1) + P(X = 2) + P(X = 4) = \frac{7}{8} + \frac{1}{8} = \frac{8}{8} = 1$$

$$F(x = 5) = P(X \le 5) = F(x = 4) + P(X = 5) = 1 + 0 = 1$$

$$\therefore F(x) \begin{cases} 0 & X < -2 \\ 1/4 & -2 \le X < 1 \\ 3/8 & 1 \le X < 2 \\ 7/8 & 2 \le X < 4 \\ 1 & 4 \le X \end{cases}$$

H.W.



Ex) A bowl contains three red chips and five blue chips . two chips are drawn at random from the bowl . Let X denote the number of red chips in sample . Find :-

1_ The (p.m.f.) of X and draw corresponding graph .

2_ The (c.p.f.) or (c.d.f.)of X and draw it

Sol)

1- the number of red chips in the sample of size

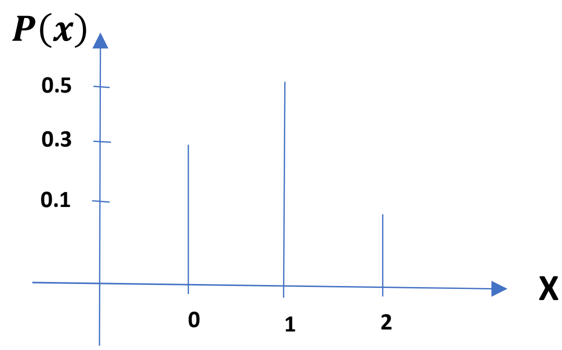
$X = 0\,,1\,,2$

$S = C_2^8 = \frac{8!}{2!\ 6!} = 28\ \ ways$

$P(X=0) = \frac{C_0^3\ C_2^5}{C_2^8} = \frac{10}{28}$, $P(X=1) = \frac{C_1^3\ C_1^5}{C_2^8} = \frac{15}{28}$

$P(X=2) = \frac{C_2^3\ C_0^5}{C_2^8} = \frac{3}{28}$

$\boldsymbol{x}$	0	1	2
$\boldsymbol{P(X=x)}$	10/28	15/28	3/28



2- The (c.p.f.) or (c.d.f.)of X

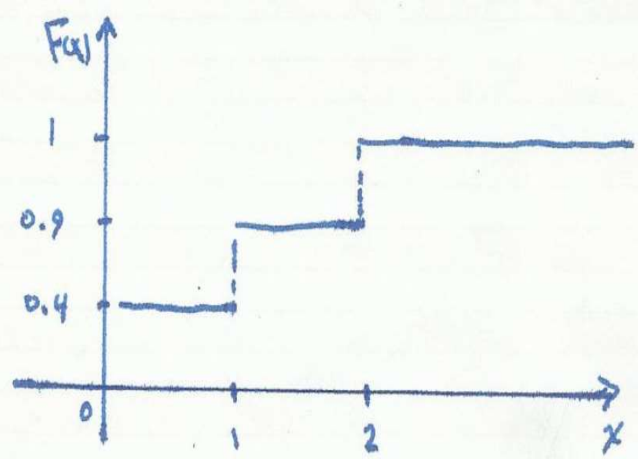
$F(x) = P(X \le x)$

$\therefore F(0) = P(X \le 0) = P(X=0) = \frac{10}{28}$

$F(1) = P(X \le 1) = P(X=0) + P(X=1) = \frac{10}{28} + \frac{15}{28} = \frac{25}{28}$

$F(2) = P(X \le 2) = P(X=0) + P(X=1) + P(X=2) = \frac{25}{28} + \frac{3}{28} = \frac{28}{28} = 1$

$$\therefore F(x) = \begin{cases} 0 & x < 0 \\ 10/28 & 0 \le x < 1 \\ 25/28 & 1 \le x < 2 \\ 1 & 2 \le x \end{cases}$$



Ex) A r.v. X has the following p.m.f.

x	-2	-1	0	1	2	3
$\boldsymbol{P(X=x)}$	K	2K	3K	5K	7K	2K

Find :

1- The value of K .

2- $P(X<1)$, $P(X\geq 0)$, $P(-1\leq X<2)$

Sol)

1- $\sum_{i=-2}^{3} P(x_i) = 1$

$\rightarrow K + 2K + 3K + 5K + 7K + 2K = 1$

$$\rightarrow 20\,K = 1 \quad \rightarrow \quad K = \frac{1}{20}$$

2- a- $P(X<1) = P(X=0) + P(X=-1) + P(X=-2)$

$$= \frac{3}{20} + \frac{2}{20} + \frac{1}{20} = \frac{6}{20}$$

b- $P(X\geq 0) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$$= \frac{3}{20} + \frac{5}{20} + \frac{7}{20} + \frac{2}{20} = \frac{17}{20}$$

c- $P(-1\leq X<2) = P(X=-1) + P(X=0) + P(X=1)$

$$= \frac{2}{20} + \frac{3}{20} + \frac{5}{20} = \frac{10}{20}$$

Note :-

$\{P(X\leq a) = 1 - P(X>a) \;\; ; \;\; P(X>a) = P(X\leq a) = 1\}$

$\{P(X\geq 0) = 1 - P(X<0) \qquad\qquad\qquad\qquad\qquad\qquad\}$

Ex) let the (P. m. f.) of the r.v. X is given by :

$$\boldsymbol{P(x)} = \begin{cases} \frac{1}{3} & ; \quad \boldsymbol{x = 0, 1, 2} \\ \boldsymbol{0} & ; \quad \textit{other wise} \end{cases}$$

1- Find (c.p.f.) or (c.d.f.) or (D.F.) or $F(X)$ and draw it :

2- Find $P(X > 0)$; $P(X \le 1)$; $P(X \le 0)$; $P(|X| < 1)$; $P(|X| + 1 < 2)$; $P(X > 1)$

<u>Sol)</u> 1- $F(X) = P(X \le x)$

$$F(0) = P(X < 0)$$

$$F(0) = P(X \le 0) = P(X = 0) = \frac{1}{3}$$

$$F(1) = P(X \le 1) = P(X = 0) + P(X = 1) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$$F(2) = P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2) = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$$

$$\therefore F(X) = \begin{cases} 0 & X < 0 \\ \frac{1}{3} & 0 \le X < 1 \\ \frac{2}{3} & 1 \le X < 2 \\ 1 & 2 \le X \end{cases}$$

2- i- $P(X > 0) = P(X = 1) + P(X = 2) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$

ii- $P(X \le 1) = P(X = 1) + P(X = 0) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$

iii- $P(X \le 0) = P(X = 0) = \frac{1}{3}$

iv- $P(|X| < 1) = P(-1 < X < 1) = P(0 \le X < 1) = P(X = 0) = \frac{1}{3}$

or $P(-1 < X < 0) + P(0 \le X < 1) = P(X = 0) = \frac{1}{3}$

v- $P(|X| + 1 < 2) = P(|X| < 1) = P(-1 < X < 1) = \frac{1}{3}$

vi- $P(X > 1) = P(X = 2) = \frac{1}{3}$

3- Probability Density Function (p.d.f.) دالة الكثافة الاحتمالية

Definition :

The probability density function of continuous r.v. X is a function (f) which the probability that the area under the curve of this function corresponding to any interval is equal to the probability that X will have a value in this interval . Then $f(X)$ is called the probability density function (p.d.f.) .

Properties :

1- $f(X) \geq 0$; $0 \leq f(X) \leq 1$

2- $\int_{-\infty}^{\infty} f(X)\, dx = 1$

3- For any interval [a,b]

$$P(a \leq x \leq b) = \int_{a}^{b} f(x)dx = F(b) - F(a)$$

$$F(X) = \int_{-\infty}^{x} f(t)dt \quad or \quad F(Y) = \int_{-\infty}^{y} f(t)dt$$

$$P(X \leq y) = P(X < y) = \int_{-\infty}^{y} f(x)dx$$

Notes :

1- $P(a \leq x \leq b) = P(a < x \leq b) = P(a \leq x < b) = P(a < x < b) = \int_{a}^{b} f(x)dx$

2- $P(X = a) = zero$ لأنه لا يوجد مساحة للمنحنى عند نقطة واحدة

3- $P(X \leq a) = P(X < a) = \int_{-\infty}^{a} f(x)dx$

4- $P(X > a) = P(X \geq a) = \int_{a}^{\infty} f(x)dx$

5- $P(X > a) = 1 - P(X \leq a)$

$P(X < a) = 1 - P(X \geq a)$

$P(X \leq a) = 1 - P(X > a)$

الملاحظة الخامسة تستخدم في حالة المتغير المستمر و/ او المتقطع

Ex) Let the r. v. X have the (p.d.f)

$$f(x) = \begin{cases} 2x & ; \quad 0 < x < 1 \\ 0 & ; \quad o.w. \end{cases}$$

Find : $P\left(\frac{1}{2} < x < \frac{3}{4}\right)$ & $P\left(-\frac{1}{2} < x < \frac{1}{2}\right)$

Sol)

1- $P\left(\frac{1}{2} < x < \frac{3}{4}\right) = \int_{\frac{1}{2}}^{\frac{3}{4}} f(x)dx = \int_{\frac{1}{2}}^{\frac{3}{4}} 2x = 2\frac{x^2}{2}\Big|_{\frac{1}{2}}^{\frac{3}{4}} = \left(\frac{3}{4}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{9}{13} - \frac{1}{4} = \frac{5}{16}$

Or $P(a < x < b) = F(b) - F(a)$

2- $P\left(-\frac{1}{2} < x < \frac{1}{2}\right) = P\left(-\frac{1}{2} < x < 0\right) + P\left(0 < x < \frac{1}{2}\right)$

تم التجزئة لكون الفترة من $0 < x < 1$ وان $-\frac{1}{2}$ تقع خارج هذه الفترة لذلك نلجئ الى التجزئة

$$\int_{-\frac{1}{2}}^{0} f(x)dx + \int_{0}^{\frac{1}{2}} f(x)dx \rightarrow \int_{-\frac{1}{2}}^{0} 2x\, dx + \int_{0}^{\frac{1}{2}} 2x\, dx$$

$$0 + 2\frac{x^2}{2}\Big|_0^{\frac{1}{2}} = \left(\frac{1}{2}\right)^2 - 0 = \frac{1}{4}$$

3- (p.d.f) هل ان الدالة هي

$\rightarrow \int_0^1 f(x)dx = \int_0^1 2x\, dx = 2\frac{x^2}{2}\Big|_0^1 = 1 - 0 = 1$ تحقق الشرط

4- $P(X \le 0.3) = P(X < 0.3) = \int_0^{0.3} 2x\, dx = 2\frac{x^2}{2}\Big|_0^{0.3} = (0.3)^2 = 0.09$

$\rightarrow P(X > 0.3) = \int_{0.3}^{1} 2x\, dx = 2\frac{x^2}{2}\Big|_{0.3}^{1} = 1 - (0.3)^2 = 1 - 0.09 = 0.91$

Or $1 - P(X \le 0.3)$

4- Distribution Function

Definition :

The distribution function of continuous random variable is denoted by $F(X)$ and is defined by :

$$F(X) = P(X \le x) = \int_{-\infty}^{x} f(t)dt$$

Properties

1_ $F(\infty) = \lim_{x \to \infty} F(x) = 1$

2_ $F(-\infty) = \lim_{x \to -\infty} F(x) = 0$

3_ $F(x_1) \leq F(x_2) \quad ; \quad if \;\; x_1 \leq x_2$

4_ $P(a \leq x \leq b) = F(b) - F(a)$

$$P(x > b) = 1 - F(b)$$

Ex) من المثال السابق

5- $F(X) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt = \int_{0}^{x} 2t\, dt = 2\frac{t^2}{2}|_0^x$

$\therefore F(X) = x^2$

$\therefore P\left(\frac{1}{2} < x < \frac{3}{4}\right) = F\left(\frac{3}{4}\right) - F\left(\frac{1}{2}\right) = \frac{9}{16} - \frac{1}{4} = \frac{5}{16}$

Ex) (H.W.) من المثال السابق جد ما يلي

1- $P\left(X < \frac{1}{2}\right) = \int_0^{\frac{1}{2}} 2x\, dx$

2- $P(X > 0.3) = \int_{0.3}^{1} 2x\, dx$

3- $P(|X| < 1) = P(-1 < X < 1) = \int_{-1}^{0} 2x\, dx + \int_{0}^{1} 2x\, dx$

4- $P(0.2 < X < 0.3) = \int_{0.2}^{0.3} 2x\, dx \;\; or \;\; F(0.3) - F(0.2)$

5- $P(-1 < X < 0.3)$

6- $P(0.4 < X < 3)$

7- $P(X = 0.5)$

$$\therefore F(X) = \begin{cases} 0 & X \leq 0 \\ x^2 & 0 < X < 1 \\ 1 & X \geq 1 \end{cases}$$

ملاحظة : في حالة وجود في السؤال فقط $F(X)$ (دالة التوزيع) والمطلوب إيجاد دالة الكثافة الاحتمالية فيتم اشتقاق الدالة للحصول على (p.d.f.)

$F'(X) = (x^2)' = 2x = f(x)$

$\therefore f(x) = \frac{\partial F(X)}{\partial X}$

Ex) Suppose that X is a continuous r.v. whose probability density function (p.d.f.) is given by :

$$f(X) = \begin{cases} c(4x - 2x^2) & 0 < X < 2 \\ 0 & o.w. \end{cases}$$

Find :

1- What is the value of the constant c ?

2- Find the probability that $X > 1$?

3- Find the probability that $X > 0$?

4- The distribution function $F(X)$?

Sol)

1- $\int_{-\infty}^{\infty} f(x)dx = 1 \rightarrow \int_0^2 c(4x - 2x^2)\, dx = 1$

$\rightarrow c[\int_0^2 4x\, dx - \int_0^2 2x^2\, dx] = 1$

$\rightarrow c\left[\frac{4\,x^2}{2}\Big|_0^2 - \frac{2\,x^3}{3}\Big|_0^2\right] = 1$

$\rightarrow c\left[(8-0) - (\frac{16}{3} - 0)\right] = 1 \rightarrow c\left[\frac{8}{3}\right] = 1 \rightarrow \therefore c = \frac{3}{8}$

$\therefore p.d.f.$

$$f(X) = \begin{cases} \left(\frac{12x}{8} - \frac{6x^2}{8}\right) & 0 < X < 2 \\ 0 & o.w. \end{cases}$$

ويجب التأكد من انها دالة كثافة احتمالية : -

$\rightarrow \int_0^2 \frac{12x}{8} - \frac{6x^2}{8}\, dx \rightarrow \frac{12}{8}\int_0^2 x\, dx - \frac{6}{8}\int_0^2 x^2\, dx$

$\rightarrow \frac{12}{8}\left(\frac{x^2}{2}\Big|_0^2 - \frac{6}{8}\,\frac{x^3}{3}\Big|_0^2\right) \Rightarrow \frac{12}{8}\left(\frac{4}{2}\right) - \frac{6}{8}\left(\frac{8}{3}\right) \Rightarrow 3 - 2 = 1$ تحقق الشرط

2- $P(X > 1) = \int_1^2 f(x)dx = \int_1^2 \frac{12}{8}x\, dx - \int_1^2 \frac{6}{8}x^2 dx =$

$\rightarrow \frac{12}{8}\left(\frac{x^2}{2}\Big|_1^2 - \frac{6}{8}\,\frac{x^3}{3}\Big|_1^2\right) = \frac{12}{8}\left(\frac{4}{2} - \frac{1}{2}\right) - \frac{6}{8}\left(\frac{8}{3} - \frac{1}{3}\right) = \frac{12}{8}\left(\frac{3}{2}\right) - \frac{6}{8}\left(\frac{7}{3}\right) = \frac{9}{4} - \frac{7}{4} = \frac{1}{2}$

Or $\quad 1 - F(1) = \frac{1}{2}$

3- $P(X > 0) = \int_0^2 f(x)dx = \frac{12}{8}\int_0^2 x\, dx - \frac{6}{8}\int_0^2 x^2 dx = 1$

4- $P(-1 < x < 1) = P(-1 < x < 0) + P(0 < x < 1)$

$$= 0 + \int_0^1 f(x)dx = \frac{1}{2}$$

Or $\quad F(1) - F(0) = \frac{1}{2}$

5- Distribution Function or $F(X)$.

$$F(X) = P(X \le x) = \int_0^x f(t)dt = \frac{12}{8}\int_0^x t\, dt - \frac{6}{8}\int_0^x t^2\, dt$$

$$\therefore F(X) = \frac{12}{8}\left(\frac{t^2}{2}\Big|_0^x - \frac{6}{8}\,\frac{t^3}{3}\Big|_0^x\right) = \frac{12}{8}\left(\frac{x^2}{2}\right) - \frac{6}{8}\left(\frac{x^3}{3}\right) = \frac{6\,x^2}{8} - \frac{2\,x^3}{8}$$

$$\therefore F(X) = \begin{cases} 0 & X < 0 \\ \frac{6\,x^2}{8} - \frac{2\,x^3}{8} & 0 \le X \le 2 \\ 1 & X \ge 2 \end{cases}$$

$\boldsymbol{X}$	0	1	2
$\boldsymbol{F(X)}$	0	½	1

Ex) let the random variable X have the probability density function .

$$f(X) = \begin{cases} cx^2 & -1 < X < 1 \\ 0 & o.w. \end{cases}$$

1- Determine the value of the constant c and $F(X)$

2- Find $P(0 < X < 1)$, $P(0 < X \le 3)$, $P\left(X = \frac{1}{2}\right)$, $P\left(-1 < X < \frac{1}{2}\right)$

Sol)

1- $\int_{-1}^1 f(x)\, dx = 1 \;\Rightarrow\; c\int_{-1}^1 x^2\, dx = 1 \Rightarrow c(\frac{x^3}{3}\Big|_{-1}^1 = 1 \Rightarrow c\left[\frac{1}{3} + \frac{1}{3}\right] = 1$

$$\Rightarrow c\left[\frac{2}{3}\right] = 1 \Rightarrow c = \frac{3}{2}$$

$$\therefore f(X) = \begin{cases} \frac{3}{2}x^2 & -1 < X < 1 \\ 0 & o.w. \end{cases}$$

And

$$F(X) = P(X \le x) = \int_{-1}^{x} f(t)dt = \int_{-1}^{x} \frac{3}{2} t^2\, dt = \frac{3}{2}\left(\frac{t^3}{3}\right|_{-1}^{x} = \frac{3}{2}\left(\frac{x^3}{3} + \frac{1}{3}\right) = \frac{1}{2}(x^3 + 1)$$

$$\therefore F(X) = \begin{cases} 0 & X < -1 \\ \frac{1}{2}(x^3+1) & -1 \le X \le 1 \\ 1 & X \ge 1 \end{cases}$$

2- i- $P(0 < X < 1) = \int_0^1 f(x)dx = \frac{3}{2}\int_0^1 x^2 dx = \frac{3}{2}\left(\frac{x^3}{3}\right|_0^1 = \frac{1}{2}$

Or $\quad F(b) - F(a) = F(1) - F(0) = 1 - \frac{1}{2} = \frac{1}{2}$

ii- $P(0 < X \le 3) = P(0 < X < 1) + P(1 < X \le 3) = \frac{1}{2} + 0 = \frac{1}{2}$

iii- $P\left(X = \frac{1}{2}\right) = zero \quad or \quad F\left(\frac{1}{2}\right) - F\left(\frac{1}{2}\right) = zero$

iv- $P\left(-1 < X < \frac{1}{2}\right) = \int_{-1}^{\frac{1}{2}} f(x)dx = \frac{3}{2}\int_{-1}^{\frac{1}{2}} x^2 dx = \frac{3}{2}\left(\frac{x^3}{3}\right|_{-1}^{\frac{1}{2}} = \frac{3}{2}\left(\frac{\left(\frac{1}{2}\right)^3}{3} - \frac{(-1)^3}{3}\right) = \frac{9}{16}$

Or $\quad F(b) - F(a) = F\left(\frac{1}{2}\right) - F(-1) = \frac{9}{16}$

Ex) Consider the distribution function

$$F(X) = \begin{cases} 0 & X = 0 \\ X & 0 \le X < 1 \\ 1 & X \ge 1 \end{cases}$$

Find : $P\left(\frac{1}{4} < X < \frac{3}{4}\right) \quad ; \quad P\left(-1 < X < \frac{1}{2}\right)$

Sol)

$$\frac{\partial F(X)}{\partial X} = f(x) = \begin{cases} 0 & for \quad x < 0 \quad and \quad x \ge 1 \\ 1 & for \quad 0 < x < 1 \end{cases}$$

i- $P\left(\frac{1}{4} < X < \frac{3}{4}\right) = \int_{\frac{1}{4}}^{\frac{3}{4}} 1\; dx = x\,\Big|_{\frac{1}{4}}^{\frac{3}{4}} = \frac{3}{4} - \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$

Or $\quad F\left(\frac{3}{4}\right) - F\left(\frac{1}{4}\right) = \frac{1}{2}$

ii- $P\left(-1 < X < \frac{1}{2}\right) = P(-1 < X < 0) + P\left(0 < X < \frac{1}{2}\right)$

$$= 0 + \int_0^{\frac{1}{2}} 1 \; dx = x \;\Big|_0^{\frac{1}{2}} = \frac{1}{2} - 0 = \frac{1}{2}$$

Or $F\left(\frac{1}{2}\right) - F(0) = \frac{1}{2}$

Ex) Suppose the continuous distribution function is

$$F(X) = \begin{cases} 1 - e^{-x} & 0 \le X \\ 0 & o.w. \end{cases}$$

By using (p.d.f.); find: 1- $P(1 < X < 3)$ 2 – $P(X > 4)$ 3- $P(X < 2)$

Sol)

$$\frac{\partial F(X)}{\partial X} = f(x) = \begin{cases} e^{-x} & 0 < X \\ 0 & o.w. \end{cases}$$

1- $P(1 < X < 3) = \int_1^3 e^{-x}\,dx = -e^{-x}\,|_1^3 = -e^{-3} + e^{-1}$

2 – $P(X > 4) = \int_4^{\infty} e^{-x}\,dx = -e^{-x}\,|_4^{\infty} = -e^{-\infty} + e^{-4} = 0 + e^{-4}$

3- $P(X < 2) = \int_{-\infty}^{2} e^{-x}\,dx = -e^{-x}\,|_{-\infty}^{2} = -e^{-2} + e^{-\infty} = -e^{-2}$

Ex) If the r.v. X has the following p.d.f.

$$f(x) = \begin{cases} k(X+1) & -1 < X < 1 \\ 0 & o.w. \end{cases}$$

Find : 1- The constant k .

2- The distribution function ($F(X)$) .

Sol)

1- $\int_{-\infty}^{\infty} f(x)\,dx = 1 \rightarrow \int_{-1}^{1} k(X+1)\,dx = 1$

$$k\left(\frac{x^2}{2} + x\right)|_{-1}^{1} \rightarrow k\left[\left(\frac{1}{2}+1\right) - \left(\frac{1}{2}-1\right)\right] = 1$$

$$\rightarrow k\left[\frac{1}{2} + 1 - \frac{1}{2} + 1\right] = 1 \rightarrow k * 2 = 1 \rightarrow k = \frac{1}{2}$$

$$\therefore f(x) = \begin{cases} \frac{1}{2}(X+1) & -1 < X < 1 \\ 0 & o.w. \end{cases}$$

2- $F(X) = \int_{-1}^{x} f(t)\,dt = \int_{-1}^{x} \frac{1}{2}(t+1)\,dt = \frac{1}{2}\left(\frac{t^2}{2} + t\right)|_{-1}^{x}$

$$= \frac{1}{2}\left[\left(\frac{x^2}{2} + x\right) - \left(\frac{1}{2} - 1\right)\right] = \frac{1}{2}\left[\frac{x^2}{2} + x + \frac{1}{2}\right]$$

$$\therefore F(X) = \begin{cases} 0 & X < -1 \\ \frac{1}{2}\left(\frac{x^2}{2} + x + \frac{1}{2}\right) & -1 \le X < 1 \\ 1 & 1 \le X \end{cases}$$

Notes :

$\frac{zero}{zero} = \ Unknown\ quantity$; $\frac{zero}{\text{قيمة}} = zero$; $\frac{zero}{\infty} = zero$; $\frac{1}{\infty} = zero$;

$\frac{\infty}{zero} = Unknown\ quantity$; $\frac{1}{zero} = Unknown\ quantity$

$e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = zero$; $e^{\infty} = \infty$; $e^{0} = 1$

$[1 + P + P^2 + P^3 + \cdots] = \frac{1}{1-P}$

$[P + P^2 + P^3 + P^4 + \cdots] = \frac{P}{1-P}$

$\left[1 + \frac{2^1}{1!} + \frac{2^{\ 2}}{2!} + \frac{2^{\ 3}}{3!} + \cdots\right] = \sum \frac{\lambda^x}{x!} = e^{\lambda}$

Ex) let the random variable X have the probability density function .

$$f(x) = \begin{cases} cxe^{-x} & 0 < X \\ 0 & o.w. \end{cases}$$

Find the constant c ?

Sol)

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \ \rightarrow \int_0^{\infty} cxe^{-x} = 1 \ \rightarrow c\int_0^{\infty} xe^{-x} = 1$$

Let $u = x \ \rightarrow du = dx$; $dv = e^{-x}dx \ \rightarrow v = -e^{-x}$

$$\int_{-\infty}^{\infty} u\,dv = uv\,|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v\,du \implies c[-x\,e^{-x}|_0^{\infty} - \int_0^{\infty} -e^{-x}\,dx\,] = 1$$

$$c[-xe^{-x}|_0^{\infty} - e^{-x}|_0^{\infty}] = 1 \ \rightarrow \ c[-(\infty e^{-\infty} - 0e^{-0}) - (e^{-\infty} - e^{-0})] = 1$$

$$\rightarrow \ c[\,-(0-0) - (0-1)] = 1 \ \rightarrow \ c = 1$$

Ex) let the random variable X have the probability density function .

$$f(x) = \begin{cases} e^{-cx} & 0 \le X \\ 0 & o.w. \end{cases}$$

Find the constant c ?

Sol)

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \;\rightarrow\; \int_{0}^{\infty} e^{-cx} dx = 1 \;\rightarrow\; -\frac{1}{c} e^{-cx}\Big|_0^{\infty} = 1$$

$$\rightarrow\; -\frac{1}{c}\,[e^{-c\infty} - e^{-c0}] = 1 \;\rightarrow -\frac{1}{c}\,(0-1) = 1 \;\rightarrow \frac{1}{c}\,(0+1) = 1 \rightarrow c = 1$$

$$\therefore f(x) = \begin{cases} e^{-x} & 0 \le X \\ 0 & o.w. \end{cases}$$

** بعض القوانين الرياضية والمتواليات المهمة :

1- $\sum_{i=1}^{n} x_i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

2- $\sum_{i=1}^{n} x_i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

3- $\sum_{i=1}^{n} x_i^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$

4- $\sum_{i=1}^{n} x_i^4 = 1^4 + 2^4 + 3^4 + \cdots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

5- $\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = \frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^3}{3!} + \cdots \cong e^{\lambda}$ حسب سلسلة تايلور

6- $\sum_{x=0}^{\infty} \lambda^x = \lambda^0 + \lambda^1 + \lambda^2 + \cdots = \frac{\text{الحد الاول}}{1-\text{الحد الثاني}} = \frac{1}{1-\lambda}$ متوالية هندسية غير منتهية

7- $\sum_{x=0}^{n} \lambda^x = \lambda^0 + \lambda^1 + \lambda^2 + \cdots + \lambda^n = \frac{1-\lambda^{n+1}}{1-\lambda}$

Ex) Let the r.v. X has the following (p.m.f.)

x	0	1	2	3	4	5	6	7
$P(X=x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

Find the constant k ?

Sol)

$\sum_{x=0}^{n} P(X = x) = 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$\rightarrow 9k + 10k^2 = 1 \ \rightarrow 10k^2 + 9k - 1 = 0 \ \rightarrow (10k - 1)(k + 1) = 0$

$\therefore \qquad 10k - 1 = 0 \quad \rightarrow \quad k = \frac{1}{10}$

$\qquad or \quad k + 1 = 0 \qquad \rightarrow k = -1$

وبما ان قيمة الثابت k هي قيمة احتمال فمن غير الممكن ان تكون قيمة سالبة وبذلك نأخذ القيمة الأولى للثابت $k = 0.1$

Ex/ $p(x) = \frac{x}{c}$; $x = 1, 2, 3, 4$

Find the constant c ? H.W.

Ex/ $p(x) = c\left(\frac{4}{5}\right)^x$; $x = 1, 2, 3, ----$

Find the constant c ? H.w.

Ex/ $P(X) = c\,\frac{4^X}{X!}$; $X = 0, 1, 2, 3, ...$

Find the Constant c ? H.w.

Ex/ H.w. :- جد قيمة C من الدالة التراكمية التالية

0	$0 < x$
C	$0 \leq x < 1$
4C	$1 \leq x < 2$
9C	$2 \leq x < 3$
16C	$3 \leq x < 4$
25C	$4 \leq x < 5$
36C	$5 \leq x < 6$
51C	$6 \leq x < 7$
68C	$7 \leq x < 8$
1	$8 \leq x$

Ex/ Let X be a r.v. having a p.m.f. / H.w.

X	-2	-1	0	2	3	5
$P(X)$	c	$2c$	$3c$	$2c$	c	$4c$

1- Find the value of c; and it's graph.
2- Find $F(x)$ and it's graph.
3- Find $P(X>2)$, $P(0<X<3)$, $P(X<-1)$

Ex/ Assuming $F(x)$ to be a distribution function of the r.v. X and is given by:-

$$F(x)=\begin{cases} 0 & X<X_1 \\ 0.5 & X_1\leq X<X_2 \\ 0.7 & X_2\leq X\leq X_3 \\ 0.8 & X_3\leq X<X_4 \\ 1 & X_4\leq X \end{cases}$$

A- Find the p.m.f. of X ?
b- Find $P(X<X_2)$, $P(X>X_4)$, $P(X_1<X<X_3)$

SoL:/

x	x_1	x_2	x_3	x_4
$p(X=x)$	0.5	0.2	0.1	0.2

$\Rightarrow \sum p(X=x)=1$

$P(X<x_2)=P(X=x_1)=0.5$

$P(X>x_4)=0$ or $1-P(X\leq x_4)=1-1=0$

or $1-F(b)=1-1=0$

$P(X_1<X<x_3)=P(X=x_2)=0.2$

Ex/ H.W

IF a r.v. X has a p.d.f.

$$f(x)=\begin{cases} \frac{3x^2}{16} & -2<X<2 \\ 0 & \text{o.w} \end{cases}$$

Find :-
1- $P(X<1)$
2- $P(|X|>1)$ $\Rightarrow 1-P(|X|\leq 1)$
3- $P(2X+3>5)$
4- $F(x)$

Ex/ A r.v. X has a p.d.f.

$$f(x) = \begin{cases} 3x^2 & 0 \le x \le 1 \\ 0 & o.w. \end{cases}$$

Find the values of a and b such that :-

a - $P(X \le a) = P(X > a)$

b - $P(X > b) = 0.1$

C - Find $F(x)$.

Sol/ $\int_{-\infty}^{\infty} f(x)\,dx = 1 \rightarrow \int_0^1 3x^2\,dx = 1 \rightarrow 3\left(\frac{x^3}{3}\right|_0^1 = 1$

$\Rightarrow 3\left(\frac{1}{3} - 0\right) = 1 \Rightarrow 1 = 1 \quad \therefore$ its a p.d.f.

a- $P(X \le a) = P(X > a) \longrightarrow \int_0^a 3x^2\,dx = \int_a^1 3x^2\,dx$

$\rightarrow \quad x^3\Big|_0^a = x^3\Big|_a^1 \Rightarrow a^3 = 1 - a^3 \Rightarrow 2a^3 = 1$

$\Rightarrow a^3 = \frac{1}{2} \Rightarrow a = \sqrt[3]{1/2}$.

b- $P(X > b) = 0.1 \rightarrow \int_b^1 3x^2\,dx = 0.1$

$\rightarrow x^3\Big|_b^1 = 0.1 \rightarrow 1 - b^3 = 0.1 \Rightarrow b^3 = 0.9 \Rightarrow b = 0.965$

C- $F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_0^x 3t^2\,dt = t^3\Big|_0^x = \underline{\underline{x^3}}$

$$\therefore F(x) = \begin{cases} 0 & x < 0 \\ x^3 & 0 \le x < 1 \\ 1 & 1 \le x \end{cases}$$

Ex/ A r.v. Y has a p.d.f. **H.W**

$$f(y) = \begin{cases} K\,y(1-y) & 0 \le y \le 1 \\ 0 & o.w. \end{cases}$$

Find:-

1- the value of K .

2 - F(y) and draw its graph.

3 - $P\left(\frac{1}{2} < y < \frac{5}{4}\right) \longrightarrow P\left(\frac{1}{2} < y \le 1\right) + P\left(1 < y < \frac{5}{4}\right)$

Chapter two

"Some Discrete Distribution"

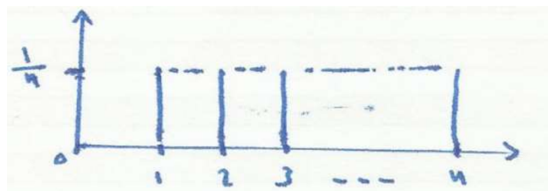
2.1- Uniform Distribution : التوزيع المنتظم

Suppose that a r.v. X takes on a finite set of real number [a , a+1 , a+2 , a+3 , … , n+a-1 = b] and have the probability mass function (p.m.f.) as given bellows :

$$P(X) = \begin{cases} \frac{1}{n} & ; \quad X = 1,2,3,\dots,n \\ 0 & \quad o.w. \end{cases}$$

Discrete uniform distribution $\quad x \sim Ud(n)$

n : it is a parameter of distribution



Discrete Uniform Distribution also has satisfied the properties of the (p.m.f.) :

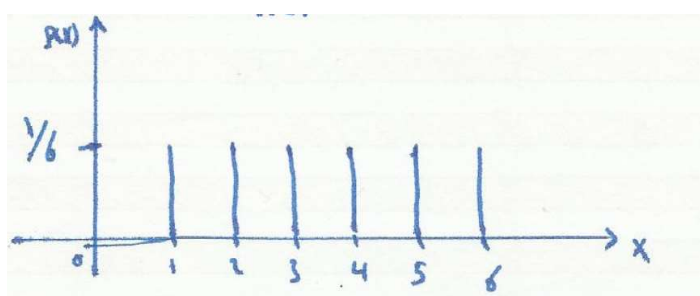
$$\sum_{x=a}^{b} P(X) = \sum_{x=a}^{b} \frac{1}{n} = 1$$

Where $\sum_{x=a}^{b} \frac{1}{n} = \frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n} = \frac{n}{n} = 1$

Ex) A dice is throw once , define a random variable X is the number shown by the dice . Find the probability mass function of X . In this case the r.v. X takes the values 1 , 2 , 3 , 4 , 5 , & 6 ; and the corresponding probability function of X will be given as :

$$P(X) = \begin{cases} \frac{1}{6} & ; \quad X = 1,2,3,4,5,6 \\ 0 & \quad o.w. \end{cases}$$

Which is clearly that $0 \le P(x) \le 1$ and $\sum_{x=1}^{n} P(x) = 1$; therefore $P(X)$ is a p.m.f. of X .



2.2- Bernoulli Distribution :

Suppose that a trial whose outcome can be classified as either a "success" or as " failure" is performed . Let X b a r.v. taking value (1) if the outcome is success and (0) if it is failure , then X is said to be a Bernoulli r.v.

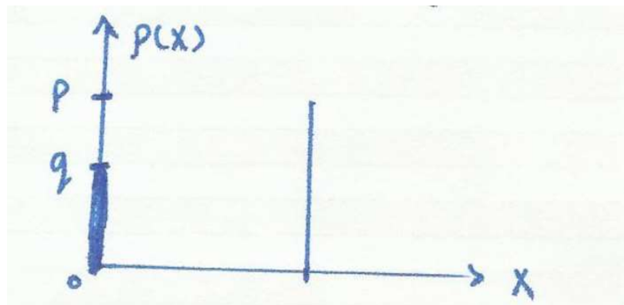
$$P(X) = \begin{cases} P^x(1-P)^{1-x} & ; \quad X = 0,1 \\ 0 & \quad o.w. \end{cases}$$

$P(X=0) = P^0 q^{1-0} = q$ فشل

$P(X=1) = P^1 q^{1-1} = P$ نجاح

Discrete Bernoulli distribution $x \sim Ber(P)$

P : it is a parameter of distribution



Ex) tossing the coins , $S = \{ H, T \}$, $\therefore P(X) = \frac{1}{2}$

$P(X) = P^x q^{1-x}$

$P(X=0) = (1/2)^0 (1/2)^{1-0} = 1/2$

$P(X=1) = (1/2)^1 (1/2)^{1-1} = 1/2$

Ex) An urn contains 10 red and 20 white balls , Draw five balls at random from the urn , one at a time and with replacement , let the draw of red ball be considered success , and the trails are independent .

Sol) $P(X=1) = {C_1^{10}}/{C_1^{30}} = \frac{10}{30} = \frac{1}{3}$ احتمال ظهور احدى الكرات حمراء (نجاح)

$$P(X) = \begin{cases} P^x q^{1-x} & ; \quad X = 0,1 \\ 0 & \quad o.w. \end{cases}$$

$P(X=0) = (1/3)^0 (2/3)^{1-0} = 2/3$

$P(X=1) = (1/3)^1 (2/3)^{1-1} = 1/3$

$\sum_{x=0}^{1} P(X) = \frac{2}{3} + \frac{1}{3} = 1 \quad \therefore \quad P.m.f.$

Ex) A fair dice is tossed one . call the outcome a success if a six is rolled , and all the other outcomes being considered failures .

This mean that we have a Bernoulli trial with probability :

$$P(X) = \frac{1}{6} \quad ; \quad S = \{ 1 ,2 ,3 ,4 ,5 ,6 \}$$

Sol)

$$P(X) = P^x q^{1-x}$$

$$P(X = x) = (1/6)^x (5/6)^{1-x}$$

$$P(X = 0) = (1/6)^0 (5/6)^{1-0} = 5/6$$

$$P(X = 1) = (1/6)^1 (5/6)^{1-1} = 1/6$$

$$\sum_{x=0}^{1} P(X) = \frac{5}{6} + \frac{1}{6} = 1 \qquad \therefore it\ is\ a\ P.m.f.$$

Bernoulli

نجاح S F فشل

محاولة ظهور حالة نجاح او فشل مرة واحدة .

اما في حالة ظهور عدد من محاولات (النجاح او الفشل) فإن :-

S F F S F F S S F S

P q q P q q P P q P

فإن X تمثل عدد محاولات النجاح (اكثر من مرة) وبذلك يستخدم توزيع ثنائي الحدين Binomial

$$P(X = x) = C_x^n \ P^x q^{n-x} \quad , \quad X = 0, 1 , 2 , 3 , \ldots , n$$ عدد محاولات النجاح

2.3- Binomial Distribution : توزيع ثنائي الحدين

Consider an experiment consisting of independent trials, each individual trial results in a success and failure with probability (P) and (1-P) respectively .

If the r.v. X represent he number of observed successes in the (n) trials , then X is Binomial random variable with the (p.m.f.) as given below .

$$P(X) = \begin{cases} C_x^n \ P^x q^{n-x} & ; \quad X = 0,1,2,\dots,n \\ 0 & \quad o.w. \end{cases}$$

And

$$\sum_{x=0}^{n} P(X) = 1 \ \rightarrow \ \sum_{x=0}^{n} C_x^n \ P^x q^{n-x} = \ (P+q)^n = 1$$

نظرية ثنائي الحدين

Discrete Binomial distribution $x \sim b(n, P)$

n, P : are the parameters of distribution

Ex) A family has 4 children, assume that the birth of each sex is equally likely. Let X denote the number of boys in family. Find the p.m.f. of X .

Sol)

We have $P(B) = P(G) = \frac{1}{2} = P \ \ and \ \ 1 - P = q \ \ , \ \ n = 4$

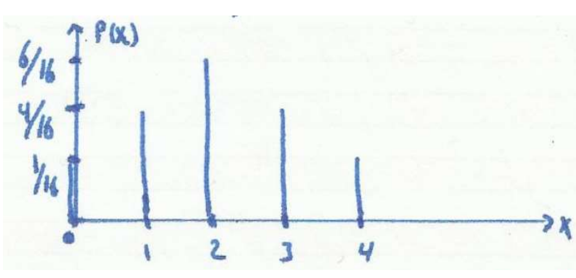
$$x \sim b\left(4, \frac{1}{2}\right) \ \ and \ \ P(X) = C_x^n \ P^x q^{n-x} \ ; \ X = 0, 1, 2, 3, 4$$

$$= C_x^4 \ (\tfrac{1}{2})^x (\tfrac{1}{2})^{4-x} = C_x^4 (\tfrac{1}{2})^4$$

The p.m.f. of X is given by:

$\boldsymbol{x}$	0	1	2	3	4
$\boldsymbol{P(X=x)}$	$1/16$	$4/16$	$6/16$	$4/16$	$1/16$

$$\boldsymbol{P(X=x)} = \frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{16} + \frac{1}{16} = \frac{16}{16} = 1 \ \text{ is p.m.f.}$$



- Find the probability that the family will have at least two boys .

$$\boldsymbol{P(X \geq 2)} = P(X = 2) + P(X = 3) + P(X = 4) = \frac{6}{16} + \frac{4}{16} + \frac{1}{16} = \frac{11}{16}$$

$$F(x) = \begin{cases} 0 & X < 0 \\ 1/16 & 0 \leq X < 1 \\ 5/16 & 1 \leq X < 2 \\ 11/16 & 2 \leq X < 3 \\ 15/16 & 3 \leq X < 4 \\ 1 & 4 \leq X \end{cases}$$

Ex) A machine produces a certain item with 0.05 defectives. random sample of 6 items is selected from the output of this machine. Let X the number of defectives in the sample. Find the probability mass function of X .

Sol) X = 0 , 1 , 2 , 3 , 4 , 5 , 6   X: تمثل عدد الوحدات التالفة او المعيبة

n = 6   حجم العينة

P = 0.05   نسبة المعيب

q = 1-P = 1 – 0.05 =   نسبة غير المعيب

$$P(X = x) = C_x^n \; P^x q^{n-x} = C_x^6 \; (0.05)^x (0.95)^{6-x} \,, \; X = 1,2,3, \dots, 6$$

The p.m.f. of X is given by:

$\boldsymbol{x}$	0	1	2	3	4	5	6
$\boldsymbol{P(X)}$	$(0.95)^6$	$6(0.05)^1(0.95)^5$	…	…	…	…	$(0.05)^6$

What is the probability that there will be 3 defective items in the sample?

$$P(X = 3) = C_3^6 \; (0.05)^3 (0.95)^{6-3}$$

مثال/ واجب : لو كان لدينا صندوق يحتوي على 30 كرة 10 حمراء و 20 بيضاء، سحبت خمسة كرات من الصندوق بشكل عشوائي، جد دالة الكتلة الاحتمالية اذا علمت ان X يمثل سحب الكرة الحمراء.

Ex) Three coins are tossed once. Let the r.v. X denoted the number of heads appears. Find the probability.

Sol)

$$S = \{\, HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

3 2 1 0

X = 0 , 1 , 2 , 3 عدد مرات ظهور الصورة

p.m.f. is given by:

$$P(X) = \begin{cases} C_x^n \; P^x q^{n-x} & ; \quad X = 0,1,2,3 \\ 0 & \quad o.w. \end{cases}$$

& $P = \frac{1}{2} = H$; $q = \frac{1}{2} = T$; $n = 3$

$\boldsymbol{x}$	0	1	2	3
$\boldsymbol{P(X)}$	1/8	3/8	3/8	1/8

* What is the probability that two head appear.

$$P(X = 2) = C_2^3 \; (\tfrac{1}{2})^2 (\tfrac{1}{2})^{3-2} = \frac{3}{8}$$

* What is the probability that at least one head appears.

$$P(X \geq 1) = 1 - P(X < 1) = 1 - P(X = 0) = 1 - \frac{1}{8} = \frac{7}{8}$$

* What is the probability that at most two head appears.

$$P(X \leq 2) = P(X = 2) + P(X = 1) + P(X = 0) = \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = \frac{7}{8}$$

Ex) Team A has probability 2/3 of wining when ever it plays, if A plays 4 games; Find the probability function that wins:

i) exactly 2 games .

ii) at least one game .

iii) more than half of the game .

Sol) $P = \frac{2}{3}$; $q = 1 - P = 1 - \frac{2}{3} = \frac{1}{3}$; $n = 4$

$$P(X) = \begin{cases} C_x^n \ P^x q^{n-x} & ; \quad X = 0,1,2,3,4 \\ 0 & \quad\quad o.w. \end{cases} \quad ; \quad X \sim b(4, \tfrac{2}{3})$$

i) $P(X = 2) = C_2^4 \ (\frac{2}{3})^2 (\frac{1}{3})^{4-2} = \frac{24}{81}$

ii) $P(X \geq 1) = 1 - P(X < 1) = 1 - P(X = 0) = 1 - C_0^4 \ \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^{4-0} = \frac{80}{81}$

iii) $P(X > 2) = P(X = 3) + P(X = 4) = \frac{48}{81}$

Ex) A study has shown that in Baghdad university 70 % of all staff own two television sets, find the probability that among 8 of each staff:-

a) 2 will have 2 Tv. Sets .

b) at least 6 will have 2 Tv. Sets .

c) at most 2 will have 2 Tv. Sets .

Sol) P = 0.70 ; q = 0.30 ; n = 8 ; $X \sim b(8, 0.70)$

$$P(X) = \begin{cases} C_x^n \ P^x q^{n-x} & ; \quad X = 0,1,2,3,\dots,8 \\ 0 & \quad\quad o.w. \end{cases}$$

a) $P(X = 2) = C_2^8 \ (0.7)^2 (0.3)^{8-2} = 0.01$

b) $P(X \geq 6) = P(X = 6) + P(X = 7) + P(X = 8) = 0.5517$

c) $P(X \leq 2) = P(X = 2) + P(X = 1) + P(X = 0) = 0.0113$

2.4- Poisson Distribution:

The Poisson distribution appears in many natural and physical phenomena suck as:

1 - The number of misprints per page in large text.

2- The number of accidents per unit of time {hour, day, week or month} on a highway.

الجسيمات المنبعثة   مشعة   مادة

3- The number of $\alpha -$ particles emitted by a radioactive substance per unit of time.

4- The number of telephone calls per unit of time received at some switchboard.

5- The number of customers entering a post office or any (government department) on a given period of time.

Always $\quad n \to \infty \quad$ كبيرة جدا

$$P(X) = \begin{cases} \frac{e^{-\lambda}\lambda^{x}}{x!} & ; \quad X = 0,1,2,3, \ldots \\ 0 & \quad o.w. \end{cases}$$

Discrete Poisson distribution $\quad x \sim Po(\lambda)$

λ : it is a parameter of distribution.

p.m.f. $\quad \sum_{x=0}^{\infty} P(X) = 1$

$$\sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^{x}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x}}{x!} = e^{-\lambda}[1 + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots] = e^{-\lambda}e^{\lambda} = e^{0} = 1$$

ويتم تحويل توزيع Binomial الى توزيع Poisson عندما $n \to \infty$ وان $P \to 0$ لتصبح معلمة التوزيع بالشكل التالي:-

$$\lambda = nP$$

وهو قانون يستخدم لاستخراج قيمة λ عند عدم توفرها في السؤال.

Ex) Suppose that 3% of the items made by a factory are defective. A sample of 100 items is drawn at random. What is the probability that it will contain exactly 2 defective items? Using

a) Binomial Distribution ; b) Poisson distribution

Sol) $P = 0.03 \quad , \quad q = 1 - P = 0.97 \quad , \quad n = 100$

a) $P(X = 2) = C_2^{100} \ (0.03)^2(0.97)^{98} = 0.2251$

b) $\lambda = nP = 100 * 0.03 = 3$

$$P(X = 2) = \frac{e^{-3}3^x}{x!} = \frac{e^{-3}3^2}{2!} = 0.22404$$

Ex) The average of claims filed against an insurance company is 2 claims per day. What is the probability that on any given day:

1- Exactly one claim is filed against the insurance company?

2- No claim is filed against the insurance company?

3- Exactly three claims are filed against the insurance company?

Sol)

1- $\lambda = 2$ { average number of claims per day }

X = 1 { exact number of claims filed }

According to the Poisson distribution:

$$P(X) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!} & ; \quad X = 0,1,2,3,\dots \\ 0 & \quad o.w. \end{cases}$$

$$P(X = 1) = \frac{e^{-2}2^1}{1!} = \frac{(2.71828)^{-2} * 2}{1} = 0.27067$$

2- $$P(X = 0) = \frac{e^{-2}2^0}{0!} = (2.71828)^{-2} = 0.13534$$

3- $$P(X = 3) = \frac{e^{-2}2^3}{3!} = \frac{8}{6 * (2.71828)^2} = 0.18045$$

Ex) A student makes, on the average, one typing error on each page, what is the probability that the student would make exactly 2 typing error in a 3-page term paper?

$n * P = \lambda = 3$ (Average number of typing errors per 3 pages)

$X = 2$ (Exact number of typing errors in a 3-page term paper)

According to the Poisson distribution:

$n * P = 3 * 1 = 3$

$\therefore\ x \sim Po(\lambda = 3)$

$$P(X) = \begin{cases} \frac{e^{-3}3^x}{x!} & ; \quad X = 0,1,2,3,\ldots \\ 0 & \quad o.w. \end{cases}$$

$$P(X=2) = \frac{e^{-3}3^2}{2!} = \frac{9}{2*(2.71828)^3} = 0.22404$$

2.5- Geometric Distribution: التوزيع الهندسي

Suppose that independent trials, each one having a probability (P) of being a success, are performed until the first success occurs. If X denotes the number of trials required, then:

يمثل حالات الفشل الى حين وصول اول حالة نجاح من عدة محاولات للفشل.

$$P(X) = \begin{cases} Pq^x & ; \quad X = 0,1,2,3,\ldots \\ 0 & \quad o.w. \end{cases} \quad ; \quad P + q = 1$$

Discrete Geometric distribution $x \sim Ge(P)$

P : it is a parameter of distribution.

To show that $P(X)$ is $P.m.f.$

متوالية هندسية غير منتهية

$$\sum_{x=0}^{\infty} P(X) = \sum_{x=0}^{\infty} Pq^x = P\sum_{x=0}^{\infty} q^x = P[1 + q + q^2 + q^3 + \cdots] = P\left[\frac{1}{1-q}\right]$$

$$= P\left[\frac{1}{P}\right] = 1$$

ملاحظة :

أحيانا تكتب الدالة بدلالة جميع المحاولات

عدد المحاولات الكلية

$$Y = X + 1$$

النجاح الوحيد

عدد المحاولات الفشل

$$\rightarrow X = Y - 1 \quad \text{where} \quad Y \sim Ge(P)$$

$$\therefore \; P(Y) = \begin{cases} Pq^{y-1} & ; \quad Y = 1,2,3,\ldots \\ 0 & \quad o.w. \end{cases}$$

Ex) If the probability that a person will believe a rumor (اشاعة) about the retirement (تقاعد) of a certain politician is 1/3 , then the probability that the fifth person to hear the rumor will be the first one to believe it is.

Sol)

$$P(X=5) = Pq^{x-1} = \frac{1}{3}(\frac{2}{3})^{5-1} = \frac{16}{243}$$

Ex) If the probability of the birth of a child with a defective heart in a special hospital is 0.07, then the probability that the 10^{th} child born in the hospital is the first one which has a defective heart is:

Sol)

$$P(X=10) = Pq^{x-1} = (0.07)(0.93)^9 = 0.036428$$

Ex) The probability that a man hit the target is 5/6.

I) If he fires 5 times, what is the probability that he hits the target 4 time?

II) What is the probability that he will hit the target for the first time at his sixth shot?

Sol)

I) $X \sim b(n=5\,, P=\frac{5}{6})$

$$P(X=x) = C_x^n\ P^x q^{n-x} = C_x^5\ (\frac{5}{6})^x(\frac{1}{6})^{5-x} \qquad X=0,1,2,3,4,5$$

$$P(X=4) = C_4^5\ (\frac{5}{6})^4(\frac{1}{6})^1 =$$

II) $X \sim Ge(P=\frac{5}{6})$

$$P(X=x) = Pq^x \qquad\qquad X=0,1,2,3,\ldots$$

$$P(X=5) = (\frac{5}{6})(\frac{1}{6})^5$$

Or

$$P(X=6) = (\frac{5}{6})(\frac{1}{6})^{6-1}$$ او بدلالة الدالة الكلية

Ex) Assume that you toss a coin until you get the first head. What is the *p.m.f.* of this experiment.

Sol) $X \sim Ge(P = \frac{1}{2})$

$$P(X) = Pq^x = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^x \quad ; \quad X = 0,1,2,3, \dots$$

What is the probability that you will get the first head at the third trial?

$X \sim Ge\left(P = \frac{1}{2}\right) \quad ; \quad X = 0,1,2,3$ تمثل محاولات الفشل

$P(X) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^x \; ; \; X = 0,1,2,3$

$P(X = 2) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2$ عدد محاولات الفشل

$P(Y) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{3-1}$ تمثل عدد المحاولات الكلية

ملاحظه:

في حالة احتواء السؤال على كلمة (first) او (until) في اغلب الأحيان فإن التوزيع هو هندسي.

Ex) Let X, Y two random variable with binomial distribution with parameter (2,P) and (3,P) respectively, if $P(X \geq 1) = 0.75$, find $P(Y \geq 1)$

Sol) $X \sim b(2\,,P) \quad , \; Y \sim b(3\,,P)$

$$P(X \geq 1) = 1 - P(X < 1)$$

$$= 1 - P(X = 0)$$

$\because \; P(X \geq 1) = 0.75$

$\rightarrow \; P(X = 0) = 1 - \; 0.75 = 0.25$

$\rightarrow \; P(X = 0) = C_0^2 \; P^0 q^{2-0} = \; q^2$

$\therefore \; q^2 = 0.25 \quad \rightarrow \quad q = \sqrt[2]{0.25} = 0.5$

$\because \; p + q = 1 \quad \rightarrow \quad P + 0.5 = 1 \; \Longrightarrow \; P = 0.5$

$\therefore \; P(Y \geq 1) = 1 - P(Y < 1) = 1 - P(Y = 0)$

$$= 1 - \; C_0^3 \,(0.5)^0 (0.5)^3 \quad = 1 - \; (0.5)^3$$

$\Longrightarrow \; P(Y \geq 1) = 0.875$

- صندوق يحتوي على 5 كرات حمراء و 6 بيضاء وسبعة سوداء، سحبت كرة من الصندوق واذا كانت الكرة حمراء تعني النجاح وان أي لون اخر يعني الفشل، ما هي الدالة الاحتمالية .$P.m.f.$ لهذا التوزيع؟

$$X \sim Ber\left(P = \frac{5}{18}\right) \quad ; \quad X = 0,1$$

$$P(X) = P^x q^{1-x}$$

- صندوق يحتوي علة 18 كرة، إذا تم سحب كرة واحدة، ما هي .$P.m.f.$ لهذا التوزيع؟

$$X \sim Un(n) \quad ; \quad X = 1,2,3, \dots ,18$$

$$P(X) = \frac{1}{n} = \frac{1}{18}$$

- صندوق يحتوي كرات سحبت 5 كرات من الصندوق علماً ان عدد الكرات الكلية هي 18 كرة ، ما هي .$P.m.f.$ لهذا التوزيع؟

$$X \sim b\left(n = 5, P = \frac{5}{18}\right)$$

واذا كانت X تمثل الكرات الحمراء فإن

$$P(X = x) = C_x^n \; P^x q^{n-x} = \; C_x^5 \; \left(\frac{5}{18}\right)^x \left(\frac{13}{18}\right)^{5-x} \qquad X = 0,1,2,3,4,5$$

- تم الحصول على اول كرة حمراء في السحبة السادسة حيث ان X تمثل حالات الفشل.

$$X \sim Ge\left(P = \frac{5}{18}\right) \quad ; \quad X = 0,1,2,3, \dots$$

$$P(X) = Pq^x$$

Ex) let $x \sim Po(\lambda)$, and $P(X = 1) = 2\,P(X = 2)$; Find the value of λ

Sol)

$$P(X) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!} & ; \quad X = 0,1,2,3, \dots \\ 0 & \quad o.w. \end{cases}$$

$$P(X = 1) = 2\,P(X = 2) \Longrightarrow \frac{e^{-\lambda}\lambda^1}{1!} = 2\;\frac{e^{-\lambda}\lambda^2}{2!}$$

$$\Longrightarrow e^{-\lambda} = \; e^{-\lambda}\lambda$$

$$\Longrightarrow \lambda = \frac{e^{-\lambda}}{e^{-\lambda}} = 1$$

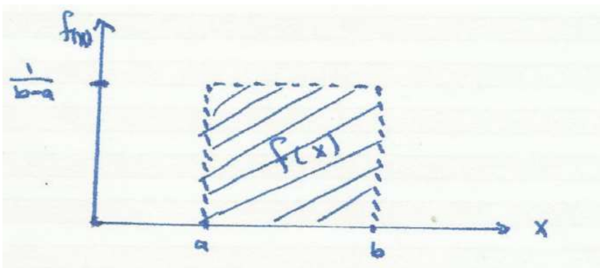
Some Continuous Distribution

3.1) Uniform Distribution:

A r.v. X is said to have uniform distribution in the interval [a,b] if its *p.d.f.* *(probability density function)* is given by:

$$f(X) = \begin{cases} \frac{1}{b-a} & a < X < b \\ 0 & o.w. \end{cases}$$

Where (a) and (b) are real constants with $a < b$. the graph of $f(X)$ is shown in



Continuous Uniform distribution $\quad x \sim Uc(a,b)$

a, b : are the parameters distribution.

ملاحظة:

التوزيع المنتظم المتقطع يحتوي على معلمة واحدة $x \sim Ud(n)$ معلمة التوزيع n ، اما التوزيع المنتظم المستمر يحتوي على معلمتين $x \sim Uc(a,b)$ معلمة التوزيع a, b علما ان معلمة التوزيع تمثل فترة $a < X < b$ للتوزيع .

To show that $f(X)$ is *p.d.f.* then: $\int_{-\infty}^{\infty} f(X)dx = 1$

$$\int_a^b \frac{1}{b-a} dx = \frac{1}{b-a} \int_a^b dx = \frac{1}{b-a} (X|_a^b = \frac{b-a}{b-a} = 1 \quad \therefore \ is \ P.d.f.$$

The distribution function (D.f) or $F(X)$ is given by:

$$F(X) = \int_{-\infty}^{x} f(t)dt$$

Then

$$F(X) = \int_a^x \frac{1}{b-a} dt = \frac{1}{b-a} \int_a^x dt = \frac{1}{b-a} (t|_a^x = \frac{x-a}{b-a}$$

$$\therefore F(X) = \begin{cases} 0 & X < a \\ \frac{x-a}{b-a} & a \le X < b \\ 1 & b \le X \end{cases}$$

Ex/

1- $f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & o.w \end{cases}$

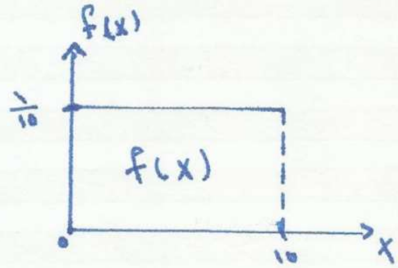
2- $f(x) = \begin{cases} \frac{1}{5} & 2 < x < 7 \\ 0 & o.w. \end{cases}$

3- $f(x) = \begin{cases} \frac{1}{4} & -1 < x < 3 \\ 0 & o.w \end{cases}$

Ex/ $X \sim UC(0,10)$

$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & o.w \end{cases}$

$f(x) = \begin{cases} \frac{1}{10} & 0 < x < 10 \\ 0 & o.w. \end{cases}$



$f(x)$ is p.d.f if $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\int_0^{10} \frac{1}{10}\,dx = \frac{1}{10}\int_0^{10} dx = \frac{1}{10}\left(x\right|_0^{10} = \frac{10}{10} = \underline{\underline{1}}$ is a p.d.f.

$F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x < b \\ 1 & b \leq x \end{cases}$ or $F(x) = \int_{-\infty}^{x} f(t)\,dt$

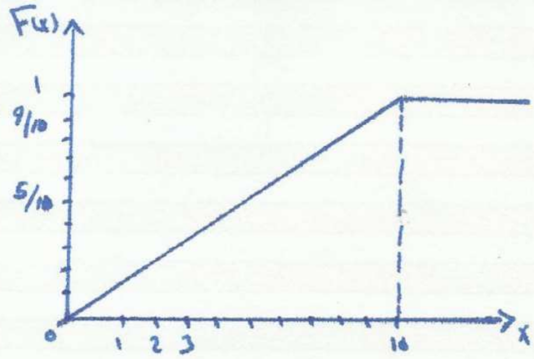
$\therefore F(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{10} & 0 \leq x < 10 \\ 1 & 10 \leq x \end{cases}$

x	$F(x) = \frac{x}{10}$
0	0
1	1/10
2	2/10
3	3/10
4	4/10
5	5/10
⋮	⋮
10	1

1- $P(X<3)=\int_0^3 f(x)\,dx$

2- $P(X>6)=\int_6^{10} f(x)\,dx$

3- $P(3<X<8)=\int_3^8 f(x)\,dx$



رسم دالة $F(x)$

4- $P(|X|<8)=P(-8<X<8)=P(-8<X<0)+P(0<X<8)$

$$= 0+\int_0^8 f(x)\,dx$$

Ex/

$X \sim Uc(-a, a)$; $P(X \leq 1)=\frac{1}{7}$; Find the constant a?

Sol/

$$f(x)=\frac{1}{b-a}=\frac{1}{a-(-a)}=\frac{1}{2a} \qquad -a<x<a$$

$\because P(X \leq 1)=\frac{1}{7}$

$$\therefore \int_{-a}^{1}\frac{1}{2a}\,dx=\frac{1}{7} \Rightarrow \frac{1}{2a}\left(x\Big|_{-a}^{1}\right)=\frac{1}{7} \Rightarrow \frac{1}{2a}(1+a)=\frac{1}{7}$$

$$\Rightarrow \frac{1}{2a}+\frac{1}{2}=\frac{1}{7} \Rightarrow \frac{1}{2a}=\frac{1}{7}-\frac{1}{2} \Rightarrow \frac{1}{2a}=\frac{-5}{14} \Rightarrow a=-\frac{14}{10}$$

Ex) Buses arrive at the bus stop at 10-minute intervals, starting at 6 Am that is arrive at { 6 , 6:10 , 6:20 , 6:30 , and so on }, if a passenger arrives at the bus stop at a time that is uniformly distributed between 6 and 6:20 , Find the probability that he waits for:

1) Less than 7 minutes for a bus?

2) More than 4 minutes for a bus?

دقائق انتظار الراكب

Sol) $\underbrace{6 \quad , \quad 6{:}10 \quad , \quad 6{:}20}$, 6:30 , …

$$f(X)=\begin{cases}\frac{1}{20} & 0<X<20\\ 0 & o.w.\end{cases}$$

1) ان المسافر او الراكب سوف ينتظر اقل من 7 دقائق

$6:3 \rightarrow 6:10 \quad or \quad 6:13 \rightarrow \ 6:20$

$P(3 < x < 10) + P(13 < x < 20)$

$$\int_3^{10} f(x)dx + \int_{13}^{20} f(x)dx = \int_3^{10} \frac{1}{20} dx + \int_{13}^{20} \frac{1}{20} dx = \frac{1}{20}[(10-3)+(13-20)]$$

$$\frac{1}{20}[7+7] = \frac{14}{20} = \frac{7}{10} = 0.7$$

2) ان المسافر او الراكب سوف ينتظر اكثر من 4 دقائق

$6:0 \rightarrow 6:4 \quad or \quad 6:10 \rightarrow \ 6:14$

$P(0 < x < 4) + P(10 < x < 14)$

$$\int_0^4 f(x)dx + \int_{10}^{14} f(x)dx = \int_0^4 \frac{1}{20} dx + \int_{10}^{14} \frac{1}{20} dx = \frac{1}{20}[(4-0)+(14-10)]$$

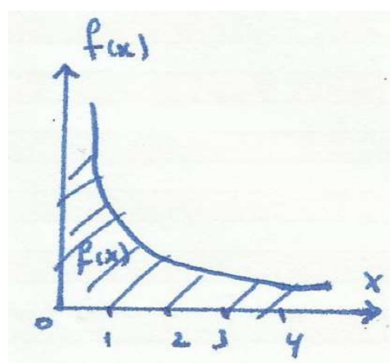
$$\frac{1}{20}[4+4] = \frac{8}{20} = \frac{4}{10} = 0.4$$

Or $\quad P(0 < x < 4) = F(b) - F(a) = F(4) - F(0)$

3.2) Exponential Distribution:

A r.v. X is said to have an exponential distribution if its *p.d.f.* (probability density function) is given by:

$$f(X) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}} & 0 \le X \\ 0 & o.w. \end{cases}$$



Or

$$f(X) = \begin{cases} \theta e^{-\theta x} & 0 \le X \\ 0 & o.w. \end{cases}$$

Where $\theta > 0$

Exponential distribution $\quad x \sim Exp(\theta)$

θ : is a parameter distribution.

To show that $f(X)$ is *p.d.f.* then: $\int_{-\infty}^{\infty} f(X)dx = 1$

$$\int_0^{\infty} \frac{1}{\theta} e^{-\frac{x}{\theta}} dx = \frac{1}{\theta}\int_0^{\infty} e^{-\frac{x}{\theta}} dx = -(e^{-\frac{x}{\theta}}|_0^{\infty} = -\left(e^{-\frac{\infty}{\theta}} - e^{-\frac{0}{\theta}}\right) = -(0-1) = 1$$

The distribution function (D.f) or $F(X)$ is given by:

$$F(X) = \int_{-\infty}^{x} f(t)dt$$

Then

$$F(X) = \int_0^x \frac{1}{\theta} e^{-\frac{t}{\theta}} dt = -(e^{-\frac{t}{\theta}}|_0^x = -\left(e^{-\frac{x}{\theta}} - 1\right) = 1 - e^{-\frac{x}{\theta}}$$

$$\therefore F(X) = \begin{cases} 0 & X < 0 \\ 1 - e^{-\frac{x}{\theta}} & 0 \le X \end{cases}$$

Or

$$F(X) = \begin{cases} 0 & X < 0 \\ 1 - e^{-x\theta} & 0 \le X \end{cases}$$

Ex) The mileage (المسافة المقطوعة) (in thousand of miles) which car owners (مالك السيارة) get with a certain kind of tyer is r.v. having an exponential with $\theta = 20$. Find the probability that one of these tyers will last (يتحمل او يدوم):

1) At most 12000 miles

2) Anywhere from 18000 to 24000 miles

Sol)

Let X denote the mileage of this tyer (in 1000 miles)

1) $P(X \le 12) = \int_0^{12} \frac{1}{20} e^{-\frac{x}{20}} dx = -(e^{-\frac{x}{20}}|_0^{12} = -\left(e^{-\frac{12}{20}} - e^{-\frac{0}{20}}\right) = 1 - e^{-0.6}$

$= 1 - 0.549 = 0.451$

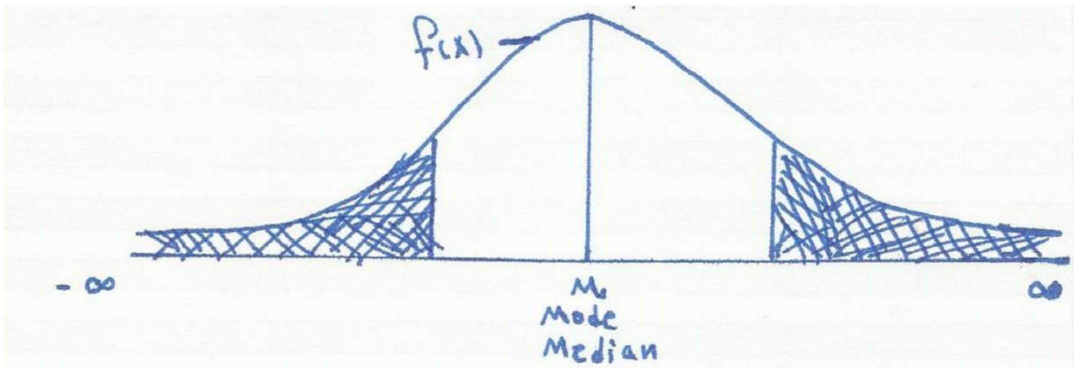
2) $P(18 < X < 24) = \int_{18}^{24} \frac{1}{20} e^{-\frac{x}{20}} dx = -(e^{-\frac{x}{20}}|_{18}^{24} = -\left(e^{-\frac{24}{20}} - e^{-\frac{18}{20}}\right)$

$= -(e^{-1.2} - e^{-0.9}) = -(0.301 - 0.406) = 0.105$

3.3) Normal Distribution:

The r.v. X is said to have an Normal distribution if its *p.d.f.* (probability density function) is given by:

$$f(X) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} & -\infty < X < \infty \\ 0 & o.w. \end{cases}$$



Normal distribution $x \sim N(\mu, \sigma^2)$

μ, σ^2 : are the parameters distribution.

Nots:

1) If $x \sim N(\mu, \sigma^2)$, then $Z = \frac{x-\mu}{\sigma}$ is standard normal variate with

$Mean = \mu = 0$ & $variance = \sigma^2 = 1$

$\therefore\ x \sim SN(0,1)$

2) the *P.d.f.* of the standard normal variate Z is given by

$$f(Z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Z^2} \qquad -\infty < Z < \infty$$

And the corresponding distribution function $\emptyset(z) = P(Z \le z)$ is given by:

$$\emptyset(z) = P(Z \le z) = \int_{-\infty}^{Z} f(u)du = \int_{-\infty}^{Z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

Or

The value of $\emptyset(z)\quad for\quad Z \geq 0$ are given in table of the standard normal distribution.

او يتم الحصول على قيمة $\emptyset(z)$ من الجداول الإحصائية للتوزيع الطبيعي القياسي.

$$P(a < Z < b) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Z^2} dz$$

Or

$$P(a < Z < b) = \emptyset(b) - \emptyset(a) \quad for \quad a < b$$

Note:

$$\emptyset(z) = P(Z \leq z)$$

$$P(a < Z < b) = \emptyset(b) - \emptyset(a)$$

$$\therefore P(Z < -a) = P(Z > a) = 1 - P(Z \leq a) = 1 - \emptyset(a) = \emptyset(-a)$$

- $P(2 < Z < 3.1) = \emptyset(3.1) - \emptyset(2) = 0.999 - 0.9772 = 0.0218$
- $P(Z \leq -0.56) = P(Z > 0.56) = 1 - P(Z \leq 0.56)$
  $= 1 - \emptyset(0.56) = 1 - 0.7123 = 0.2877$

If $x \sim N(\mu, \sigma^2)$, the distribution function of X can be expressed as:

$$F(a) = P(X \leq a)$$

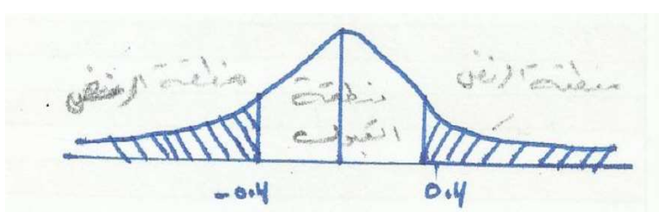
$$\therefore F(a) = P(X \leq a) = P(\frac{x-\mu}{\sigma} < \frac{a-\mu}{\sigma})$$

$$= P\left(Z < \frac{a-\mu}{\sigma}\right) = \emptyset(\frac{a-\mu}{\sigma})$$

Ex) $x \sim N(4,25)$

1) $P(X < 6) = P\left(\frac{x-\mu}{\sigma} < \frac{6-\mu}{\sigma}\right) = P\left(Z < \frac{6-4}{5}\right) = P\left(Z < \frac{2}{5}\right) = P(Z < 0.04)$

$= \emptyset(0.4) = 0.554$



2) $P(3 < X < 5) = P\left(\frac{3-\mu}{\sigma} < \frac{x-\mu}{\sigma} < \frac{5-\mu}{\sigma}\right)$

$$= P\left(\frac{3-4}{5} < Z < \frac{5-4}{5}\right) = P(-0.2 < Z < 0.2)$$

$$= \emptyset(0.2) - \emptyset(-0.2)$$

$$= \emptyset(0.2) - [1 - \emptyset(0.2)]$$

$$= 0.5793 - [1 - 0.5793] = 0.1586$$

Ex) $x \sim N(3,9)$

1) $P(X > 0) = 1 - P(X \leq 0) = 1 - P(\frac{x-\mu}{\sigma} < \frac{0-\mu}{\sigma})$

$$= 1 - P\left(Z < \frac{0-3}{3}\right) = 1 - P(Z < -1) = 1 - \emptyset(-1)$$

$$= 1 - [1 - \emptyset(1)] = 1 - 1 + \emptyset(1) = \emptyset(1) = 0.84134$$

2) $P(|X-3| > 6) = 1 - P(|X-3| \leq 6) = 1 - P(-6 < X - 3 < 6)$

$$= 1 - P(-6 + 3 < X - 3 + 3 < 6 + 3)$$

$$= 1 - P(-3 < X < 9) = 1 - P\left(\frac{-3-\mu}{\sigma} < \frac{x-\mu}{\sigma} < \frac{9-\mu}{\sigma}\right)$$

$$= 1 - P\left(\frac{-3-3}{3} < Z < \frac{9-3}{3}\right) = 1 - P(-2 < Z < 2)$$

$$= 1 - [\,\emptyset(2) - \emptyset(-2)] = 1 - [\,\emptyset(2) - (1 - \emptyset(2))]$$

Ex) $x \sim N(25,36)$;

$P(|X - 25| \leq c) = 0.9544$; Find the value of c.

Sol)

$P(-c \leq X - 25 \leq c) = 0.9544 \quad \rightarrow \quad P(-c + 25 \leq X \leq c + 25) = 0.9544$

$\rightarrow \quad P\left(\frac{(-c+25)-\mu}{\sigma} \leq \frac{x-\mu}{\sigma} \leq \frac{(c+25)-\mu}{\sigma}\right) = 0.9544$

$\rightarrow \quad P\left(\frac{(-c+25)-25}{6} \leq Z \leq \frac{(c+25)-25}{6}\right) = 0.9544$

$$\rightarrow \quad P\left(\frac{-c}{6} \leq Z \leq \frac{c}{6}\right) = 0.9544$$

$$\because P(a < X < b) = \emptyset(b) - \emptyset(a)$$

$$\therefore \emptyset\left(\frac{c}{6}\right) - \emptyset\left(\frac{-c}{6}\right) = 0.9544 \quad \rightarrow \quad \emptyset\left(\frac{c}{6}\right) - \left[1 - \emptyset\left(\frac{c}{6}\right)\right] = 0.9544$$

$$\rightarrow 2\emptyset\left(\frac{c}{6}\right) - 1 = 0.9544 \quad \rightarrow 2\emptyset\left(\frac{c}{6}\right) = 1 + 0.9544 \quad \rightarrow \emptyset\left(\frac{c}{6}\right) = \frac{1.9544}{2}$$

$$\rightarrow \emptyset\left(\frac{c}{6}\right) = 0.9772 = \emptyset(2)$$

$$\rightarrow \emptyset\left(\frac{c}{6}\right) = \emptyset(2) \quad \rightarrow \frac{c}{6} = 2 \quad \Longrightarrow c = 12$$

ملاحظة:

يمكن تحويل توزيع ثنائي الحدين الى التوزيع الطبيعي ايضاً كما هو الحال في توزيع بواسون المتقطع لإيجاد المعدل (λ) وذلك عندما

$$n \rightarrow \infty \quad \& \quad P \rightarrow 0$$

ويمكن الحصول على معلمات التوزيع الطبيعي عند توفر توزيع ثنائي الحدين بالمعلمات n, P

$$\{\mu = n\,P \quad ; \quad \sigma^2 = nPq\,\}$$

$$x \sim B(n, P) \xrightarrow[P \rightarrow 0]{n \rightarrow \infty} x \sim N(\,\mu = n\,P,\ \sigma^2 = nPq\,)$$

Where

$$Z = \frac{x - nP}{\sqrt{nPq}}$$

Ex) A fair coin is tossed 20 times. Determine the probability that the number of heads are:

1- 14

2- Between 14 and 16 inclusive, by using:

I) Binomial distribution.

II) Normal approximation to the Binomial distribution.

Sol)

I) $x \sim B\left(20\,,\frac{1}{2}\right)$

1- $P(X=14) = C_{14}^{20}\ (\frac{1}{2})^{14}(\frac{1}{2})^{6} = 0.3696$

2- $P(14 \le X \le 16) = P(X=14) + P(X=15) + P(X=16)$

$$= C_{14}^{20}\ (\frac{1}{2})^{14}(\frac{1}{2})^{6} + C_{15}^{20}\ (\frac{1}{2})^{15}(\frac{1}{2})^{5} + C_{16}^{20}\ (\frac{1}{2})^{16}(\frac{1}{2})^{4}$$

$$= 0.0563$$

II) $x \sim N(\ \mu = n\,P,\ \sigma^2 = nPq\ )$

$x \sim N(\ 20 * \frac{1}{2},\ 20 * \frac{1}{2} * \frac{1}{2}\ )$

$x \sim N(\ \mu = 10,\ \sigma^2 = 5\ )$

1- $P(X=14) = P(13.5 \le X \le 14.5)$

$$= P\left(\frac{13.5-\mu}{\sigma} < \frac{x-\mu}{\sigma} < \frac{14.5-\mu}{\sigma}\right)$$

$$= P\left(\frac{13.5-10}{\sqrt{5}} < Z < \frac{14.5-10}{\sqrt{5}}\right)$$

$$= P(1.57 < Z < 2.01)$$

$$= \emptyset(2.01) - \emptyset(1.57) = 0.032$$

2- $P(14 \le X \le 16) = P(13.5 \le X \le 16.5)$

$$= P\left(\frac{13.5-10}{\sqrt{5}} < Z < \frac{16.5-10}{\sqrt{5}}\right)$$

$$= P(1.55 < Z < 2.9)$$

$$= \emptyset(2.9) - \emptyset(1.55) = 0.0564$$

Generating Function

الدالة المولدة

4.1) Mathematical Expectation: التوقع الرياضي

Expectation of a Discrete Random Variable

* Definition:

The Expectation of a discrete random variable X having a *p.m.f. P(X)* , is denoted be:

$$E(X) = \sum X_i \ P(X_i)$$

Properties:

1- If X is a discrete r.v. with *p.m.f. P(X)* ; then for any real – valued function

$$Y = g(X)$$

$$E(Y) = E\big(g(X)\big) = \sum g(X) \ P(X_i)$$

2- $E(c) = c$

3- $E(cx) = cE(x)$

4- $E(x \mp c) = E(x) \mp c$

5- $E(cx \mp by) = cE(x) \mp bE(y)$

ملاحظة:

يمكن ان تأخذ قيمة التوقع قيمة موجبة او سالبة $-\infty < E(X) < \infty$

Expectation of a Continuous Random Variable

* Definition:

The Expectation of a continuous random variable with *p.d.f. f(x)* , is denoted be:

$$E(X) = \int_{-\infty}^{\infty} x \ f(x)\, dx$$

Let *X* be a r.v. with *p.d.f. f(x)* in the interval [a,b], then:

$$E(X) = E\big(g(X)\big) = \int_a^b E\big(g(X)\big)\, f(x)\, dx$$

Ex)

$$P(X) = \begin{cases} \frac{1}{6} & X = 1,2,3,4,5,6 \\ 0 & o.w. \end{cases}$$

Find the following:

$$E(X),\quad E(X^2),\ E(X^4),\ E\left(\frac{1}{X}\right),\ E(X+4),\ E(2X-1)^2$$

Sol)

$$E(X) = \sum_{x=1}^{6} X_i P(X_i) = \left[1 * \frac{1}{6} + 2 * \frac{1}{6} + 3 * \frac{1}{6} + \cdots + 6 * \frac{1}{6}\right]$$

$$= \frac{1}{6}[1 + 2 + 3 + \cdots + 6] = \frac{1}{6} * \frac{6(6+1)}{2} = \frac{7}{2}$$

Or $$= \frac{1}{6} * 21 = \frac{7}{2}$$

$$g(x) = X^2$$

$$E(X^2) = \sum_{x=1}^{6} g(x)P(x) = \sum_{x=1}^{6} X^2 \frac{1}{6} = \frac{1}{6}\sum_{x=1}^{6} X^2$$

$$= \frac{1}{6} * \frac{6(6+1)(2*6+1)}{6} = \frac{91}{6} = 15.1667$$

$$E(X^4) = \sum_{x=1}^{6} X^4 P(x) = \sum_{x=1}^{6} X^4 \frac{1}{6} = \frac{1}{6}\left[\frac{6(6+1)(2*6+1)(3*6+3*6-1)}{30}\right]$$

$$E\left(\frac{1}{X}\right) = \sum_{x=1}^{6} \frac{1}{X} P(x) = \sum_{x=1}^{6} \frac{1}{X} * \frac{1}{6} = \frac{1}{6}\left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{6}\right]$$

And

$$E\left(\frac{1}{X}\right) \neq \frac{1}{E(x)}$$

$$E(X+4) = E(X) + 4 = \frac{7}{2} + 4 = \frac{15}{2}$$

$$E(2X-1)^2 = \sum_{x=1}^{6} (2X-1)^2 P(x) =$$

Or

$$E(4X^2 - 4X + 1) = 4E(X^2) - 4E(X) + 1$$

$$= 4 * \left(\frac{91}{6}\right) - 4 * (\frac{7}{2}) + 1$$

Ex)

$$f(X) = \begin{cases} \frac{1}{2} & 2 < X < 4 \\ 0 & o.w. \end{cases}$$

Find :

$$E(X), \quad E(X^2), \quad E\left(\frac{1}{X}\right), \quad E(X+1), \quad E(2X-1)^2$$

Sol)

$$E(X) = \int_{-\infty}^{\infty} X\, f(x)dx = \int_{2}^{4} X\, \frac{1}{2}\, dx = \frac{1}{2}(X^2|_2^4 = \frac{1}{2}(6) = 3$$

$$E(X^2) = \int_{-\infty}^{\infty} X^2\, f(x)dx = \int_{2}^{4} X^2\, \frac{1}{2}\, dx = \frac{1}{2}(\frac{X^3}{3}|_2^4 = \frac{56}{6}$$

$$E\left(\frac{1}{X}\right) = \int_{-\infty}^{\infty} \frac{1}{X}\, f(x)dx = \int_{2}^{4} \frac{1}{X}\,\frac{1}{2}\, dx = \frac{1}{2}(Ln\, X|_2^4 = ** \neq \frac{1}{E(X)}$$

$$E(X+1) = E(X) + 1 = 3 + 1 = 4$$

$$E(2X-1)^2 = E(4X^2 - 4X + 1) = 4E(X^2) - 4E(X) + 1$$

$$= 4 * \left(\frac{56}{6}\right) - 4 * (3) + 1 = 63.67$$

Ex) A dice is thrown once, where X will represent the money that a person wins or loses, as follows:

عدد ارقام الزار	$P(x)$	عدد النقود التي يربحها او يخسرها
1	1/6	−1
2	1/6	+2
3	1/6	+3
4	1/6	−4
5	1/6	+3
6	1/6	−6

قيمة سالبة (خسارة) $E(x)=$

قيمة موجبة (ربح) $E(x)=$

صفر (عدم وجود ربح او خسارة) $E(x)=$

Sol)

$$E(X) = \sum_{x=1}^{6} XP(X) = \left[\frac{-1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{-4}{6} + \frac{3}{6} + \frac{-6}{6}\right] = -\frac{1}{2}$$ خسارة

Ex) A box contains 4 red, 6 white and 8 green balls. If a red ball is drawn, he will win 3 dinars, but if a white ball is drawn, he will win 2 dinars, note that the person expected that he would get nothing, i.e., $E(X) = 0$, what is the value of k If he is drawn a green ball?

صندوق يحتوي على 4 كرات حمراء ، 6 كرات بيضاء ، 8 كرات خضراء ، فإذا تم سحب كرة حمراء سوف يربح 3 دنانير اما إذا سحب كرة بيضاء سوف يربح دينارين ، علما ان الشخص توقع عدم حصوله على أي شيء أي ان $E(X) = 0$ ، فما هي قيمة k اذا سحب كرة خضراء ؟

	عدد الدنانير	$P(x)$
حمراء	3	4/18
بيضاء	2	6/18
خضراء	K	8/18

$$\therefore E(x) = \sum x\,p(x)$$

$$0 = 3 \times \frac{4}{18} + 2 \times \frac{6}{18} + K \times \frac{8}{18}$$

$$\therefore K = -3$$

Ex) If we have :

$$f(X) = \begin{cases} 6X(1-X) & 0 < X < 1 \\ 0 & o.w. \end{cases}$$

Find : $E(X)$, $E\left(\frac{1}{1-X}\right)$

Sol)

$$E(X) = \int_{-\infty}^{\infty} X\, f(x) dx = \int_{0}^{1} X\, 6X(1-X)\, dx = \int_{0}^{1} 6(X^2 - X^3)\, dx =$$

$$= 6(\frac{X^3}{3} - \frac{X^4}{4}|_0^1 = 6\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{2}$$

$$E\left(\frac{1}{1-X}\right) = \int_{-\infty}^{\infty} \frac{1}{1-X}\, f(x) dx = \int_{0}^{1} \frac{1}{1-X}\, 6X(1-X)\, dx = \int_{0}^{1} 6X\, dx =$$

$$= 6(\frac{X^2}{2}|_0^1 = 6\left(\frac{1}{2}\right) = 3$$

4.2) **Variance:** **التباين**

Definition :

The Variance of a r.v. X is denoted by $V(x)\ or\ \sigma^2(x)$ and is defined by:

$$V(x) = \sigma^2(x) = E[X - E(X)]^2 = E(X^2) - [E(X)]^2$$

$$\therefore E(X) = \mu_x = mean$$

$$or\ E(Y) = \mu_y$$

$\sigma(x) = \sqrt{\sigma^2(x)}$ standard deviation

Proof:

$$V(x) = E(X - E(X))^2$$

$$\because E(X) = \mu_x$$

$$\therefore V(x) = E(X - \mu_x)^2 = E(X^2 - 2\mu_x X + {\mu_x}^2)$$

$$V(x) = E(X^2) - 2\mu_x E(X) + {\mu_x}^2$$

$$V(x) = E(X^2) - 2{\mu_x}^2 + {\mu_x}^2 = E(X^2) - {\mu_x}^2$$

$$= E(X^2) - (E(X))^2$$

Where

$$E(x) = \sum_{\forall x} x\,p(x) \qquad E(x^2) = \sum_{\forall x} x^2\,p(x)$$

$$E(x) = \int_{-\infty}^{\infty} x\,f(x)\,dx \qquad E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

We know that $\sigma_x^2 \geq 0$

$\therefore\ E(X^2) \geq (E(X))^2$

كمية ثابتة موجبة

$$\sigma_x^2 = E(X^2) - (E(X))^2$$

$$A = E(X^2) - (E(X))^2$$

$$\therefore\ A + (E(X))^2 = E(X^2)$$

$$\therefore\ E(X^2) \geq (E(X))^2$$

And we can write

$$E(X^2) = E[X(X-1) + X]$$

$$= E\big(X(X-1)\big) - E(X)$$

Ex)

$$P(X) = \begin{cases} \frac{1}{6} & X = 1,2,3 \\ 0 & o.w. \end{cases}$$

Find: $V(x)$, $E(x)$

Sol)

$$E(x) = \sum_{\forall x} X\,P(X) = 1 * \frac{1}{6} + 2 * \frac{1}{6} + 3 * \frac{1}{6} = \frac{1}{6}(1+2+3) = \frac{1}{6} * 6 = 1$$

$$V(x) = E(X^2) - (E(X))^2$$

$$\therefore E(X^2) = \sum_{\forall x} X^2\,P(X) = 1^2 * \frac{1}{6} + 2^2 * \frac{1}{6} + 3^2 * \frac{1}{6} = \frac{1}{6}(1+4+9) = \frac{14}{6}$$

$$\therefore V(x) = \frac{14}{6} - (1)^2 = \frac{14}{6} - 1 = \frac{8}{6}$$

Ex)

$$f(X) = \begin{cases} \frac{1}{2} & 0 < X < 1 \\ 0 & o.w. \end{cases}$$

Find: $V(x)$

Sol)

$$E(X) = \int_{-\infty}^{\infty} X\, f(x)dx = \int_{0}^{1} X\frac{1}{2}\, dx = \frac{1}{2}(\frac{X^2}{2}|_0^1 = \frac{1}{2}\left(\frac{1}{2} - 0\right) = \frac{1}{4}$$

$$E(X^2) = \int_{-\infty}^{\infty} X^2\, f(x)dx = \int_{0}^{1} X^2\frac{1}{2}\, dx = \frac{1}{2}(\frac{X^3}{3}|_0^1 = \frac{1}{2}\left(\frac{1}{3} - 0\right) = \frac{1}{6}$$

$$\therefore V(x) = E(X^2) - (E(X))^2 = \frac{1}{6} - (\frac{1}{4})^2 = \frac{1}{6} - \frac{1}{16} = \frac{10}{96}$$

Properties of variance

1- $V(c) = 0$ where c = constant

2- $V(cx) = c^2 V(x)$

3- $V(x \mp c) = V(x)$

4- $V(cx \mp b) = c^2 V(x)$

5- $V(z) = 1$ where

$$z = \frac{x-\mu}{\sigma} \quad \rightarrow \quad V(z) = V(\frac{x-\mu}{\sigma})$$

$$= \frac{1}{\sigma^2} V(x-\mu) = \frac{1}{\sigma^2} V(x)$$

$$= \frac{1}{\sigma^2} * \sigma^2 = 1$$

4.3) Moments:

The r-th moment about the origin is denoted m_r , if it exists it is defined as:

Discrete:

$$m_r = E(X^r) = \sum_{\forall x} X^r P(X) \, ; \qquad\qquad r = 1,2,3,\ldots$$

Or

Continuous:

$$m_r = E(X^r) = \int_{-\infty}^{\infty} X^r f(x)dx$$

The r-th central moment about the mean $m = E(X)$, is denoted by μ_r , if it exists it is defined as:

Discrete:

$$\mu_r = E(X-m)^r = \sum_{\forall x} (X-m)^r P(X) \, ; \qquad\qquad r = 1,2,3,\ldots$$

Or

Continuous:

$$\mu_r = E(X-m)^r = \int_{-\infty}^{\infty} (X-m)^r f(x)dx$$

Note that:

If $r = 2 , \; then \;\; \mu_2 = E(X-m)^2 = E(X-E(X))^2 = Var(X)$

The standard deviation of X is denoted by σ , and is defined as: $\sigma = \sqrt{Var(X)}$

Ex) Let X be a r.v. with a *p.m.f.* given by:

$\boldsymbol{x}$	0	1	2	3
$\boldsymbol{P(X = x)}$	1/10	2/10	3/10	4/10

Find: $m_1 , m_2 , m_4 , 4 - th \; central \; moment \;\; \mu_4 = E(X-m)^4$

Sol)

$$r=1 \quad , \quad m_1 = E(X) = \sum_{x=0}^{3} XP(X) = \left(0 * \frac{1}{10}\right) + \left(1 * \frac{2}{10}\right) + \left(2 * \frac{3}{10}\right) + \left(3 * \frac{4}{10}\right)$$

$$= \frac{20}{10} = 2$$

$$r=2 \quad , \quad m_2 = E(X^2)$$

$$= \sum_{x=0}^{3} X^2 P(X) = \left(0^2 * \frac{1}{10}\right) + \left(1^2 * \frac{2}{10}\right) + \left(2^2 * \frac{3}{10}\right) + \left(3^2 * \frac{4}{10}\right)$$

$$= \frac{50}{10} = 5$$

$$r=4 \quad , \quad m_4 = E(X^4)$$

$$= \sum_{x=0}^{3} X^4 P(X) = \left(0^4 * \frac{1}{10}\right) + \left(1^4 * \frac{2}{10}\right) + \left(2^4 * \frac{3}{10}\right) + \left(3^4 * \frac{4}{10}\right)$$

$$= \frac{374}{10} = 37.4$$

$$\therefore \textit{the } 4-\textit{th central moment about the mean } m_1 = 2 = m$$

$$\therefore \ \mu_4 = E(X-m)^4 = E(X-2)^4 = \sum_{x=0}^{3} (X-2)^4 P(X)$$

$$= (0-2)^4\left(\frac{1}{10}\right) + (1-2)^4\left(\frac{2}{10}\right) + (2-2)^4\left(\frac{3}{10}\right) + (3-2)^4\left(\frac{4}{10}\right) = \frac{22}{10} = 2.2$$

Ex) Let X be a r.v. with a density function:

$$f(X) = \begin{cases} \frac{1}{b} & 0 < X < b \\ 0 & o.w. \end{cases}$$

Find: m_1, m_2, $Var(X)$, the 3-rd central moment and the 3-rd moment.

$$r=1 \quad , \quad m_1 = E(X) = \int_{-\infty}^{\infty} Xf(x)dx = \int_{0}^{b} X * \frac{1}{b}\, dx = \frac{1}{b}\left(\frac{X^2}{2}\Big|_0^b\right.$$

$$= \frac{1}{b}\left(\frac{b^2}{2} - 0\right) = \frac{b}{2}$$

$$r = 2 \quad , \quad m_2 = E(X^2) = \int_{-\infty}^{\infty} X^2 f(x)dx = \int_0^b X^2 * \frac{1}{b}\, dx = \frac{1}{b}(\frac{X^3}{3} |_0^b$$

$$= \frac{1}{b}(\frac{b^3}{3} - 0) = \frac{b^2}{3}$$

$$Var(X) = E\left(X - E(X)\right)^2 = E(X^2) - (E(X))^2 = m_2 - m_1^2$$

$$= \frac{b^2}{3} - (\frac{b}{2})^2 = \frac{b^2}{3} - \frac{b^2}{4} = \frac{4b^2 - 3b^2}{12} = \frac{b^2}{12}$$

$$\mu_3 = E(X - m)^3 = E\left(X - \frac{b}{2}\right)^3 = \int_0^b \left(X - \frac{b}{2}\right)^3 * \frac{1}{b}\, dx = \frac{\left(X - \frac{b}{2}\right)^4}{4b} |_0^b$$

$$= \frac{1}{4b}\left[\frac{b^4}{16} - \frac{b^4}{16}\right] = 0$$

$$r = 3 \quad , \quad m_3 = E(X^3) = \int_{-\infty}^{\infty} X^3 f(x)dx = \int_0^b X^3 * \frac{1}{b}\, dx = \frac{1}{b}(\frac{X^4}{4} |_0^b$$

$$= \frac{1}{b}(\frac{b^4}{4} - 0) = \frac{b^3}{4}$$

4.4) Moment Generating function: **الدالة المولدة للعزوم**

The moment generating function (m.g.f.) denoted by $m_x(t)$ for a r.v. X is defined to be the expectation of the exponential function e^{tx} , where t = all real numbers.

Where

$$m_x(t) = E(e^{tx}) = \sum_{\forall x} e^{tx} P(X)$$

If X is discrete r.v. with *p.m.f.* $P(X)$

Or

$$m_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

If X is continuous r.v. with *p.d.f.* $f(X)$

We call $m_x(t)$ the moment generating function because all of the moments of X can be obtained by successively على التوالي differentiating اشتقاق جزئي $m_x(t)$ and evaluating the result at $(t = 0)$.

Theorem:

If $m_x(t)$ is (*m.g.f.*) of X and if m_k is will be defined:

$$m_x^k(0) = \frac{d^k}{d_t^k} m_k(t)|_{t=0} = m_k$$

Or the relation between (*m.g.f*) and moment , we know that

$m_x(t) = E(e^{tx})$

$m_x'(t) = E(xe^{tx})$

$m_x'(0) = E(xe^0)$, $t = 0$

Hence

$$m_x'(0) = E(x * 1) = E(x) = m_1 = mean$$

- $m_x''(t) = E(x^2 e^{tx})$

  $m_x''(0) = E(x^2 e^0) = E(x^2) = m_2$

- $m_x'''(t) = E(x^3 e^{tx})$

  $m_x'''(0) = E(x^3 e^0) = E(x^3) = m_3$

In general, the k-th derivative of $m_x(t)$ is given by:

$m_x^k(t) = E(x^k e^{tx})$

$m_x^k(0) = E(x^k) = m_k$, k-th moment.

Another way to see the derivative of $m_x(t)$ is by taylor series expansion of e^{tx}:

$$m_x(t) = E(e^{tx}) = E\left(\sum_{i=0}^{\infty} \frac{(tx)^i}{i!}\right)$$

$$= E\left(1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \frac{(tx)^4}{4!} + \cdots\right)$$

$$= \left[1 + tE(x) + t^2\,\frac{E(x^2)}{2!} + t^3\,\frac{E(x^3)}{3!} + t^4\,\frac{E(x^4)}{4!} + \cdots\right]$$

$$= \left[1 + tm_1 + t^2\,\frac{m_2}{2!} + t^3\,\frac{m_3}{3!} + t^4\,\frac{m_4}{4!} + \cdots\right]$$

$$\therefore m_x'(t) = m_1 + \frac{2t\,m_2}{2!} + \frac{3t^2\,m_3}{3!} + \frac{4t^3 m_4}{4!} + \cdots$$

$$m_x'(0) = m_1 + 0 + 0 + 0 + \cdots$$

$\therefore m_x'(0) = m_1$ first moment.

$$m_x''(t) = m_2 + \frac{6t\,m_3}{3!} + \frac{12t^2\,m_4}{4!} + \frac{20t^3 m_5}{5!} + \cdots$$

$$m_x''(0) = m_2 + 0 + 0 + 0 + \cdots$$

$\therefore m_x''(0) = m_2$ Second moment

It is easy to see that:

$m_x^k(0) = m_k$, k-th moment.

Ex) Find the *m.g.f.* of X when

$$f(X) = \begin{cases} \frac{1}{\theta} & 0 < X < \theta \\ 0 & o.w. \end{cases}$$

Sol)

$$m_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx = \int_0^{\theta} e^{tx}\frac{1}{\theta}dx = \frac{1}{\theta t}\int_0^{\theta} te^{tx}dx = \frac{1}{\theta t}(\,e^{tx}|_0^{\theta}$$

$$= \frac{1}{\theta t}\left(e^{t\theta} - e^0\right) = \frac{1}{\theta t}\left(e^{t\theta} - 1\right) \quad m.g.f.$$

$$m_x'(t) = m_1 = E(x) \;\; ; \;\; m_x''(t) = m_2 = E(x^2)$$

$\therefore Var(X) = E\big(X - E(X)\big)^2 = E(X^2) - (E(X))^2 = m_2 - m_1^2$

Ex) A fair coin is flipped twice, let X be the number of head that occur, then:

$S = \{HH, HT, TH, TT\}$ and:

x	0	1	2
$P(X = x)$	$1/4$	$2/4$	$1/4$

1-find the moment generating function of X .

2- find the μ_x $and\ \sigma_x^2$

Sol)

1- $m_x(t) = E(e^{tx}) = \sum_{\forall x} e^{tx} P(X)$

$$= e^{t0} * \frac{1}{4} + e^{t1} * \frac{2}{4} + e^{t2} * \frac{1}{4}$$

$\therefore m_x(t) = \frac{1}{4} + \frac{1}{2}e^t + \frac{1}{4}e^{2t}$

2- $m_x'(t) = 0 + \frac{1}{2}e^t + \frac{2}{4}e^{2t} = \frac{1}{2}e^t + \frac{1}{2}e^{2t}$

$m_x'(0) = \frac{1}{2}e^0 + \frac{1}{2}e^{2*0} = \frac{1}{2} + \frac{1}{2} = 1$

$\therefore m_x'(0) = 1 = m_1 = E(x) = \mu_x$

$m_x''(t) = \frac{1}{2}e^t + \frac{2}{2}e^{2t} = \frac{1}{2}e^t + e^{2t}$

$\therefore m_x''(0) = \frac{1}{2}e^0 + e^{2*0} = \frac{1}{2} + 1 = \frac{3}{2}$

$\therefore m_x''(0) = \frac{3}{2} = m_2 = E(x^2)$

$\therefore \sigma_x^2 = E(x^2) - (E(x))^2 = m_2 - m_1^2$

$\therefore \sigma_x^2 = \frac{3}{2} - 1^2 = \frac{3}{2} - 1 = \frac{1}{2}$